

STRONGLY DAMPED WAVE EQUATION WITH EXPONENTIAL NONLINEARITIES

AZER KHANMAMEDOV

ABSTRACT. In this paper, we study the initial boundary value problem for the two dimensional strong damped wave equation with exponentially growing source and damping terms. We first show the well-posedness of this problem and then prove the existence of the global attractor in $(H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)$.

1. INTRODUCTION

The paper is devoted to the study of the strongly damped wave equation

$$w_{tt} - \Delta w_t + f(w_t) - \Delta w + g(w) = h. \quad (1.1)$$

The semilinear strongly damped wave equations are quite interesting from a physical viewpoint. For example, they arise in the modeling of the flow of viscoelastic fluids (see [1, 2]) as well as in the theory of heat conduction (see [3, 4]). One of the most important problems regarding these equations is to analyse their long-time dynamics in terms of attractors. The attractors for such equations have intensively been studied by many authors under different types of hypotheses. We refer to [5-7] and references therein for strongly damped wave equations with the linear damping and subcritical source term. In the critical source term case, the existence of the attractors for the strongly damped wave equations with the linear damping was proved in [8] and later in [9]. The regularity of the attractor, established in [8, 9], was proved in [10], for the critical source term case. Later in [11], it was shown that the attractor of the strongly damped wave equation with the critical source term, indeed, attracts every bounded subset of $H_0^1(\Omega) \times L^2(\Omega)$ in the norm of $H_0^1(\Omega) \times H_0^1(\Omega)$. In [12], the authors proved the existence and regularity of the uniform attractor for the nonautonomous strongly damped wave with the critical source term. The attractors for the strongly damped wave equations with the source term like polynomial of arbitrary degree were investigated in [13]. In the nonlinear subcritical damping term case, the attractors for the strongly damped wave equations were studied in [14] and [15]. In [16], the authors investigated the attractors of the abstract second order evolution equation with the damping term depending both on displacement and velocity. In particular, the results obtained in [16] can be applied to the strongly damped wave equation with subcritical nonlinearities. Attractors for strongly damped wave equations with the critical displacement dependent damping and source terms were established in [17]. Recently, in [18] the authors have proved the existence of the attractors for the equation (1.1), when the source term g is subcritical and nonmonotone damping term f is critical. Later in [19], they have improved this result for the case when both of f and g are critical and $\inf_{x \in R} f'(x) > -\lambda_1$, where λ_1 is the first eigenvalue of Laplace operator.

Although in the most of papers mentioned above, the three dimensional strongly damped equations were studied, their methods, due to the embedding $H^1(\Omega) \subset L^p(\Omega)$, $1 \leq p < \infty$, are valid also for the two dimensional equation (1.1), with polynomially (of arbitrary degree) growing f and g . The goal of the present paper is to study the two dimensional equation (1.1) with exponentially growing damping and source terms. However, in this case, the embedding $H^1(\Omega) \subset L^p(\Omega)$, $1 \leq p < \infty$, is not sufficient to control the energy integrals of (1.1) by the norm of initial data from $H_0^1(\Omega) \times L^2(\Omega)$. Therefore, we consider the equation (1.1) in the space $(H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)$ instead of the usual

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phase space $H_0^1(\Omega) \times L^2(\Omega)$. On the other hand, it arises another difficulty as the $L^\infty(\Omega)$ -estimate for the weak solutions. To overcome this difficulty, we reduce the strongly damped wave equation to the heat equation and use the regularity property of the latter (see Lemma 3.2-3.3).

The paper is organized as follows. In the next section, we state the problem and main results. In section 3, we first prove the existence of the weak solution and then establish its L^∞ regularity, which plays a key role for the uniqueness of the weak solution. After uniqueness, we show L^2 regularity for w_{tt} and then continuous dependence of the weak solution on initial data. In section 4, we first establish the dissipativity, in particular the global boundedness of solutions in $L^\infty(\Omega)$ uniformly with respect to the initial data from a bounded subset of $(H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)$, and then the asymptotic compactness, which, together with the existence of the strict Lyapunov function, lead to the existence of the global attractor. Finally, in the last section, we give some auxiliary lemmas.

2. STATEMENT OF THE PROBLEM AND RESULTS

We consider the following initial-boundary value problem:

$$\begin{cases} w_{tt} - \Delta w_t + f(w_t) - \Delta w + g(w) = h(x) & \text{in } (0, \infty) \times \Omega, \\ w = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ w(0, \cdot) = w_0, \quad w_t(0, \cdot) = w_1 & \text{in } \Omega, \end{cases} \quad (2.1)$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, $h \in L^2(\Omega)$ and the nonlinear functions f, g satisfy the following conditions:

$$\bullet \quad f, g \in C^1(\mathbb{R}), \quad f(0) = g(0) = 0, \quad \inf_{s \in \mathbb{R}} f'(s) > -\lambda_1, \quad \liminf_{|s| \rightarrow \infty} g'(s) > -\lambda_1, \quad (2.2)$$

$$\bullet \quad |f(-s)| \leq c(1 + |f(s)|), \quad \forall s \in \mathbb{R}, \quad (2.3)$$

$$\bullet \quad \int_{-\infty}^{\infty} \frac{|f'(s)|}{s(f(s) + \lambda_1 s) + 1} ds < \infty, \quad \int_{-\infty}^{\infty} \frac{|g'(s)|}{|s(g(s) + \lambda_1 s)| + 1} ds < \infty, \quad (2.4)$$

where $\lambda_1 = \inf_{\varphi \in H_0^1(\Omega), \varphi \neq 0} \frac{\|\nabla \varphi\|_{L^2(\Omega)}^2}{\|\varphi\|_{L^2(\Omega)}^2}$.

It is easy to verify that for $\alpha, \beta \in [0, 1)$ the functions $f(s) = se^{|s|^\alpha}$ and $g(s) = se^{|s|^\beta} + P_n(s)$ satisfy the conditions (2.2)-(2.4), where $P_n(s)$ is a polynomial of degree n .

Definition 2.1. The function $w \in C([0, T]; H_0^1(\Omega))$ satisfying $w_t \in C_s(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$, $f(w_t) \in L^1(0, T; L^1(\Omega))$, $g(w) \in L^1(0, T; L^1(\Omega))$, $w(0, x) = w_0(x)$, $w_t(0, x) = w_1(x)$ and the equation

$$\frac{d}{dt} \langle w_t, v \rangle + \langle \nabla w_t, v \rangle + \langle \nabla w, v \rangle + \langle f(w_t), v \rangle + \langle g(w), v \rangle = \langle h, v \rangle$$

in the sense of distributions on $(0, T)$, for all $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$, is called the weak solution to the problem (2.1) on $[0, T] \times \Omega$, where $C_s(0, T; L^2(\Omega)) = \{u : u \in L^\infty(0, T; L^2(\Omega)), \langle u, \varphi \rangle \in C[0, T], \text{ for every } \varphi \in L^2(\Omega)\}$ and $\langle \psi, \varphi \rangle = \int_{\Omega} \psi(x) \varphi(x) dx$.

Our first result is the following well-posedness theorem:

Theorem 2.1. Assume that the conditions (2.2)-(2.4) are satisfied. Then for every $T > 0$ and $(w_0, w_1) \in (H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)$, the problem (2.1) admits a unique weak solution which satisfies

$$w \in C([0, T]; H_0^1(\Omega) \cap L^\infty(\Omega)), \quad w_t \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)), \quad w_{tt} \in L_{loc}^2(0, T; L^2(\Omega))$$

and the inequalities

$$\begin{cases} \|w(t)\|_{H_0^1(\Omega)} + \|w_t(t)\|_{L^2(\Omega)} + \int_0^t \|\nabla w_t(\tau)\|_{L^2(\Omega)}^2 d\tau \\ + \int_0^t \langle f(w_t(\tau)), w_t(\tau) \rangle d\tau \leq c_1(r(w_0, w_1)), \\ \|w(t)\|_{L^\infty(\Omega)} \leq c_2(T, r(w_0, w_1)) \end{cases}, \quad \forall t \in [0, T]. \quad (2.5)$$

Moreover, if $v \in C([0, T]; H_0^1(\Omega) \cap L^\infty(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \cap W^{1,2}(0, T; H_0^1(\Omega)) \cap W_{loc}^{2,2}(0, T; L^2(\Omega))$ is also a weak solution to (2.1) with initial data $(v_0, v_1) \in (H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)$, then

$$\begin{aligned} & \|w(t) - v(t)\|_{H_0^1(\Omega)} + \|w_t(t) - v_t(t)\|_{L^2(\Omega)} \leq \\ & \leq c_3(T, \tilde{r}) \left(\|w_0 - v_0\|_{H_0^1(\Omega)} + \|w_1 - v_1\|_{L^2(\Omega)} \right), \quad \forall t \in [0, T], \end{aligned} \quad (2.6)$$

where $c_1 : R_+ \rightarrow R_+$ and $c_i : R_+ \times R_+ \rightarrow R_+$ ($i = 2, 3$) are nondecreasing functions with respect to each variable, $r(w_0, w_1) = \|(w_0, w_1)\|_{(H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)}$ and $\tilde{r} = \max\{r(w_0, w_1), r(v_0, v_1)\}$.

Hence, by Theorem 2.1, the solution operator $S(t)(w_0, w_1) = (w(t), w_t(t))$ of the problem (2.1) generates a weakly continuous (in the sense, if $\varphi_n \rightarrow \varphi$ strongly then $S(t)\varphi_n \rightarrow S(t)\varphi$ weakly star) semigroup in $(H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)$.

Definition 2.2. ([20]) Let $\{V(t)\}_{t \geq 0}$ be a semigroup on a metric space (X, d) . A compact set $A \subset X$ is called a global attractor for the semigroup $\{V(t)\}_{t \geq 0}$ iff

- A is invariant, i.e. $V(t)A = A$, $\forall t \geq 0$;
- $\lim_{t \rightarrow \infty} \sup_{v \in B} \inf_{u \in A} d(V(t)v, u) = 0$ for each bounded set $B \subset X$.

Our second result is the following theorem:

Theorem 2.2. Under conditions (2.2)-(2.4) the semigroup $\{S(t)\}_{t \geq 0}$ generated by the problem (2.1) possesses a global attractor in $(H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)$.

3. WELL-POSEDNESS

We start with the following existence result:

Lemma 3.1. Assume that the conditions (2.2)-(2.3) are satisfied. Then for every $(w_0, w_1) \in (H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)$, the problem (2.1) admits a weak solution on $[0, 1] \times \Omega$ such that

$$\lim_{t \searrow 0} \|w_t(t) - w_1\|_{L^2(\Omega)} = 0. \quad (3.1)$$

Proof. By using Galerkin's method, let us to construct approximate solutions of (2.1). Let $\{\varphi_j\}_{j=1}^\infty$ be a basis of $H^2(\Omega) \cap H_0^1(\Omega)$ consisting of the eigenfunctions of the Dirichlet problem

$$\begin{cases} -\Delta \varphi_j = \lambda_j \varphi_j & \text{in } \Omega \\ \varphi_j = 0 & \text{on } \partial\Omega \end{cases}, \quad j = 1, 2, \dots$$

According to Lemma A.1 and the embedding $H^2(\Omega) \subset C(\overline{\Omega})$, for $(w_0, w_1) \in (H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)$, there exist the sequences $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ such that

$$\begin{cases} \sum_{k=1}^n \alpha_k \varphi_k \rightarrow w_0 \text{ in } H_0^1(\Omega), \quad \sum_{k=1}^n \beta_k \varphi_k \rightarrow w_1 \text{ in } L^2(\Omega) \text{ as } n \rightarrow \infty, \\ \sup_n \left\| \sum_{k=1}^n \alpha_k \varphi_k \right\|_{C(\overline{\Omega})} \leq \|w_0\|_{L^\infty(\Omega)}. \end{cases} \quad (3.2)$$

We define the approximate solution $w_n(t)$ in the form

$$w_n(t) = \sum_{k=1}^n c_{nk}(t) \varphi_k,$$

where $c_{nk}(t)$ are determined by the system of second order ordinary differential equations

$$\begin{aligned} & \left\langle \sum_{k=1}^n c'_{nk}(t) \varphi_k, \varphi_j \right\rangle + \left\langle \sum_{k=1}^n c'_{nk}(t) \nabla \varphi_k, \nabla \varphi_j \right\rangle + \left\langle f \left(\sum_{k=1}^n c'_{nk}(t) \varphi_k \right), \varphi_j \right\rangle \\ & + \left\langle \sum_{k=1}^n c_{nk}(t) \nabla \varphi_k, \nabla \varphi_j \right\rangle + \left\langle g \left(\sum_{k=1}^n c_{nk}(t) \varphi_k \right), \varphi_j \right\rangle = \langle h, \varphi_j \rangle, \quad j = 1, 2, \dots, n, \end{aligned} \quad (3.3)$$

with the initial data

$$c_{nj}(0) = \alpha_j, \quad c'_{nj}(0) = \beta_j, \quad j = 1, 2, \dots, n. \quad (3.4)$$

Since $\det(\langle \varphi_j, \varphi_k \rangle) \neq 0$ and the nonlinear functions f and g are continuous, by the Peano existence theorem, there exists at least one local solution to (3.3)-(3.4) in the interval $[0, T_n)$. Hence this allows to construct the approximate solution $w_n(t)$. Multiplying the equation (3.3)_j by the function $c'_{nj}(t)$, summing from $j = 1$ to n and integrating over $(0, t)$, we have

$$\begin{aligned} & \frac{1}{2} \|w_{nt}(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla w_n(t)\|_{L^2(\Omega)}^2 + \langle G(w_n(t)), 1 \rangle + \int_0^t \|\nabla w_{nt}(\tau)\|_{L^2(\Omega)}^2 d\tau \\ & + \int_0^t \langle f(w_{nt}(\tau)), w_{nt}(\tau) \rangle d\tau = \frac{1}{2} \|w_{nt}(0)\|_{L^2(\Omega)}^2 + \langle h, w_n(t) \rangle \\ & + \frac{1}{2} \|\nabla w_n(0)\|_{L^2(\Omega)}^2 + \langle G(w_n(0)), 1 \rangle - \langle h, w_n(0) \rangle, \quad \forall t \in [0, T_n), \end{aligned} \quad (3.5)$$

where $G(w) = \int_0^w g(u) du$. Taking into account (2.2) and (3.2) in (3.5), we get

$$\left\{ \begin{array}{l} \|w_n(t)\|_{H^1(\Omega)}^2 \leq c, \\ \|w_{nt}(t)\|_{L^2(\Omega)}^2 \leq c, \\ \int_0^t \|w_{nt}(s)\|_{H^1(\Omega)}^2 ds \leq c, \\ \int_0^t \langle f(w_{nt}(\tau)) + \lambda_1 w_{nt}(\tau), w_{nt}(\tau) \rangle d\tau \leq c \end{array} \right. , \quad \forall t \in [0, T_n).$$

where the constant c depends on $\|(w_0, w_1)\|_{(H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)}$ and is independent of n and t . Hence, we can extend the approximate solution to the interval $[0, \infty)$ and

$$\begin{aligned} & \|w_n(t)\|_{H^1(\Omega)}^2 + \|w_{nt}(t)\|_{L^2(\Omega)}^2 + \int_0^t \|w_{nt}(s)\|_{H^1(\Omega)}^2 ds \\ & + \int_0^t \langle f_1(w_{nt}(\tau)), w_{nt}(\tau) \rangle d\tau \leq 4c, \quad \forall t \geq 0, \end{aligned} \quad (3.6)$$

where $f_1(s) = f(s) + \lambda_1 s$.

Multiplying the equation (3.3)_j by the function $c_{nj}(t)$, summing from $j = 1$ to n and integrating over $(0, t)$, we obtain

$$\begin{aligned} & \langle w_{nt}(t), w_n(t) \rangle + \frac{1}{2} \|\nabla w_n(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla w_n(s)\|_{L^2(\Omega)}^2 ds \\ & + \int_0^t \langle f(w_{nt}(s)), w_n(s) \rangle ds + \int_0^t \langle g(w_n(s)), w_n(s) \rangle ds = \langle w_{nt}(0), w_n(0) \rangle \\ & + \int_0^t \|w_{nt}(s)\|_{L^2(\Omega)}^2 ds + \frac{1}{2} \|\nabla w_n(0)\|_{L^2(\Omega)}^2 + \int_0^t \langle h, w_n(s) \rangle ds, \quad \forall t \geq 0, \end{aligned}$$

which, together with (3.6), yields

$$\int_0^t \langle g_1(w_n(s)), w_n(s) \rangle ds + \int_0^t \langle f_1(w_{nt}(s)), w_n(s) \rangle ds \leq c_1(1+t), \quad \forall t \geq 0,$$

where $g_1(s) = g(s) + \mu_1 s$ and $\mu_1 = \left| \inf_{s \in R} g'(s) \right| + 1$. Applying Lemma A.6 with choice $M = 1$ to the second integral on the left hand side of the last inequality, we find

$$\begin{aligned} & \int_0^t \langle g_1(w_n(s)), w_n(s) \rangle ds \leq c_1(1+t) \\ & + \left(\|w_n(0)\|_{L^\infty(\Omega)} + 1 \right) \int_0^1 \langle f_1(w_{nt}(s)), w_{nt}(s) \rangle ds + \|f_1\|_{C[-1,1]} \|w_n(0)\|_{L^1(\Omega)} \\ & + \int_0^1 \langle |f_1(-w_{nt}(s))|, |w_{nt}(s)| \rangle ds, \quad \forall t \in [0, 1], \end{aligned}$$

which, together with (2.3), (3.2) and (3.6), gives us

$$\int_0^1 \langle g_1(w_n(s)), w_n(s) \rangle ds \leq c_2, \quad (3.7)$$

where the constant c_2 , as c and c_1 , depends on $\|(w_0, w_1)\|_{(H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)}$ and is independent of n .

Now, multiplying the equation $(3.3)_j$, by the function $\frac{1}{\lambda_j^2} c''_{nj}(t)$ and then summing from $j = 1$ to n , we get

$$\begin{aligned} & \|(-\Delta)^{-1} w_{nt}(t)\|_{L^2(\Omega)}^2 \\ & = \langle w_{nt}(t) + w_n(t) - (-\Delta)^{-1} f(w_{nt}(t)) - (-\Delta)^{-1} g(w_n(t)) + (-\Delta)^{-1} h, (-\Delta)^{-1} w_{nt}(t) \rangle, \end{aligned}$$

and consequently

$$\begin{aligned} \|w_{nt}(t)\|_V & \leq c_3 \left(\|w_{nt}(t)\|_{L^2(\Omega)} + \|w_n(t)\|_{L^2(\Omega)} \right. \\ & \left. + \|h\|_{L^2(\Omega)} + \|f(w_{nt}(t))\|_V + \|g(w_n(t))\|_V \right), \quad \forall t \geq 0, \end{aligned} \quad (3.8)$$

where V is the dual of $H^2(\Omega) \cap H_0^1(\Omega)$. Since, by the embedding $L^1(\Omega) \subset V$,

$$\begin{aligned} \|f(w_{nt}(t))\|_V & \leq c_4 \|f(w_{nt}(t))\|_{L^1(\Omega)} \leq c_4 \|f\|_{C[-1,1]} \text{mes}(\Omega) \\ & + c_4 \int_{\{x: x \in \Omega, |w_{nt}(t, x)| > 1\}} |f(w_{nt}(t, x))| |w_{nt}(t, x)| dx \leq c_4 \|f\|_{C[-1,1]} \text{mes}(\Omega) \\ & + c_4 \langle f_1(w_{nt}(t)), w_{nt}(t) \rangle + c_5 \|w_{nt}(t)\|_{L^2(\Omega)}^2 \\ & \leq c_6 (1 + \|w_{nt}(t)\|_{L^2(\Omega)}^2 + \langle f_1(w_{nt}(t)), w_{nt}(t) \rangle) \end{aligned}$$

and similarly

$$\|g(w_n(t))\|_V \leq c_7 (1 + \|w_n(t)\|_{L^2(\Omega)}^2 + \langle g_1(w_n(t)), w_n(t) \rangle)$$

by (3.6)-(3.8), we have

$$\int_0^1 \|w_{nt}(s)\|_V ds \leq c_8, \quad (3.9)$$

where c_8 is independent of n . Thus it follows from the estimates (3.6) and (3.9) that the sequences $\{w_n\}_{n=1}^\infty$, $\{w_{nt}\}_{n=1}^\infty$ and $\{w_{ntt}\}_{n=1}^\infty$ are bounded in $L^\infty(0, 1; H_0^1(\Omega))$, $L^\infty(0, 1; L^2(\Omega)) \cap L^2(0, 1; H_0^1(\Omega))$ and $L^1(0, 1; V)$, respectively. Applying the compact embedding theorem (see [21, Corollary 4]), we obtain that the sequences $\{w_n\}_{n=1}^\infty$ and $\{w_{nt}\}_{n=1}^\infty$ are precompact in $L^2(0, 1; L^2(\Omega))$.

As a consequence, there exists a subsequence of $\{w_n\}_{n=1}^\infty$ (still denoted by $\{w_n\}_{n=1}^\infty$) and a function $w \in L^\infty(0, 1; H_0^1(\Omega))$ with $w_t \in L^\infty(0, 1; L^2(\Omega)) \cap L^2(0, 1; H_0^1(\Omega))$ such that

$$\begin{cases} w_n \rightarrow w \text{ weakly star in } L^\infty(0, 1; H_0^1(\Omega)), \\ w_n \rightarrow w \text{ strongly in } C([0, 1]; L^2(\Omega)), \\ w_{nt} \rightarrow w_t \text{ weakly star in } L^\infty(0, 1; L^2(\Omega)), \\ w_{nt} \rightarrow w_t \text{ weakly in } L^2(0, 1; H_0^1(\Omega)), \\ w_{nt} \rightarrow w_t \text{ strongly in } L^2(0, 1; L^2(\Omega)), \\ w_n \rightarrow w \text{ a.e. in } (0, 1) \times \Omega, \\ w_{nt} \rightarrow w_t \text{ a.e. in } (0, 1) \times \Omega. \end{cases} \quad (3.10)$$

Now, applying Lemma A.2, by (3.7) and (3.10)₆, we get

$$g_1(w_n) \rightarrow g_1(w) \text{ strongly in } L^1(0, 1; L^1(\Omega)),$$

which, together with (3.10)₂, implies

$$g(w_n) \rightarrow g(w) \text{ strongly in } L^1(0, 1; L^1(\Omega)). \quad (3.11)$$

Similarly, by (3.6), (3.10)₅ and (3.10)₇, we find

$$f(w_{nt}) \rightarrow f(w_t) \text{ strongly in } L^1(0, 1; L^1(\Omega)). \quad (3.12)$$

Thus, considering the last approximations and passing to the limit in (3.3), we obtain

$$\frac{d}{dt} \langle w_t, \varphi_j \rangle + \langle \nabla w_t, \nabla \varphi_j \rangle + \langle \nabla w, \nabla \varphi_j \rangle + \langle f(w_t), \varphi_j \rangle + \langle g(w), \varphi_j \rangle = \langle h, \varphi_j \rangle, \quad j = 1, 2, \dots \quad (3.13)$$

in the sense of distributions on $(0, 1)$. Since $\{\varphi_j\}_{j=1}^\infty$ is base in $H^2(\Omega) \cap H_0^1(\Omega)$, by (3.2), for every $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$, there exists $\{\alpha_{nj}\}_{j=1}^{k_n}$ such that

$$\sum_{j=1}^{k_n} \alpha_{nj} \varphi_j \rightarrow v \text{ in } H_0^1(\Omega), \text{ as } n \rightarrow \infty, \quad \text{and} \quad \sup_n \left\| \sum_{j=1}^{k_n} \alpha_{nj} \varphi_j \right\|_{C(\bar{\Omega})} \leq \|v\|_{L^\infty(\Omega)},$$

which, together with (3.13), yields that

$$\frac{d}{dt} \langle w_t, v \rangle + \langle \nabla w_t, \nabla v \rangle + \langle \nabla w, \nabla v \rangle + \langle f(w_t), v \rangle + \langle g(w), v \rangle = \langle h, v \rangle,$$

in the sense of distributions on $(0, 1)$, for every $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$. From this equation, it is easy to see that $w_{tt} \in L^1(0, 1; H^{-1}(\Omega) + L^1(\Omega))$ and

$$w_{tt} - \Delta w_t - \Delta w + f(w_t) + g(w) = h \text{ in } L^1(0, 1; H^{-1}(\Omega) + L^1(\Omega)).$$

Now, by $w \in L^\infty(0, 1; H_0^1(\Omega))$, $w_t \in L^2(0, 1; H_0^1(\Omega))$ and $w_{tt} \in L^1(0, 1; H^{-1}(\Omega) + L^1(\Omega))$, it follows that $w \in C([0, 1]; H_0^1(\Omega))$ and $w_t \in C([0, 1]; H^{-1}(\Omega) + L^1(\Omega))$. Further, applying [22, Lemma 8.1, p. 275], by $w_t \in L^\infty(0, 1; L^2(\Omega))$ and $w_t \in C([0, 1]; H^{-1}(\Omega) + L^1(\Omega))$, we have $w_t \in C_s(0, 1; L^2(\Omega))$.

By (3.10)₁ and (3.10)₄, it follows that

$$w_n \rightarrow w \text{ weakly in } C([0, 1]; H_0^1(\Omega))$$

and particularly

$$w_n(0) \rightarrow w(0) \text{ weakly in } H_0^1(\Omega),$$

which, together with (3.2)₁, yields that $w(0) = w_0$. Also, by (3.3), (3.10)-(3.12) and (3.13), we find

$$\frac{d}{dt} \langle w_{nt}, \varphi_j \rangle \rightarrow \frac{d}{dt} \langle w_t, \varphi_j \rangle \text{ weakly in } L^1(0, 1), \quad j = 1, 2, \dots$$

The last approximation, using (3.10)₃, gives us

$$\langle w_{nt}(0), \varphi_j \rangle \rightarrow \langle w_t(0), \varphi_j \rangle, \quad j = 1, 2, \dots$$

Hence, by (3.2)₁ and (3.4), we have $w_t(0) = w_1$. Thus, the function $w \in C([0, 1]; H_0^1(\Omega))$ with $w_t \in C_s(0, 1; L^2(\Omega))$ is a weak solution to (2.1) on $[0, 1] \times \Omega$.

Now, let us prove (3.1). By (3.2), (3.4) and Lebesgue's convergence theorem, it is easy to see that

$$\lim_{n \rightarrow \infty} \langle F(w_n(0)), 1 \rangle = \langle F(w_0), 1 \rangle.$$

Also, by (2.2), (3.10) and Fatou's lemma, we have

$$\liminf_{n \rightarrow \infty} \langle G(w_n(t)), 1 \rangle \geq \langle G(w(t)), 1 \rangle$$

and

$$\liminf_{n \rightarrow \infty} \int_0^t \langle f(w_{nt}(\tau)), w_{nt}(\tau) \rangle d\tau \geq \int_0^t \langle f(w_t(\tau)), w_t(\tau) \rangle d\tau.$$

Hence, passing to the limit in (3.5) and taking the weak lower semi-continuity of the norm into consideration leads to

$$\begin{aligned} & \frac{1}{2} \|w_t(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla w(t)\|_{L^2(\Omega)}^2 + \langle G(w(t)), 1 \rangle + \int_0^t \|\nabla w_t(\tau)\|_{L^2(\Omega)}^2 d\tau \\ & + \int_0^t \langle f(w_t(\tau)), w_t(\tau) \rangle d\tau \leq \frac{1}{2} \|w_1\|_{L^2(\Omega)}^2 + \langle h, w(t) \rangle \\ & + \frac{1}{2} \|\nabla w_0\|_{L^2(\Omega)}^2 + \langle G(w_0), 1 \rangle - \langle h, w_0 \rangle, \quad \forall t \in [0, 1]. \end{aligned} \quad (3.14)$$

Since $w \in C([0, 1]; H_0^1(\Omega))$, passing to the limit in (3.14) as $t \searrow 0$, we get

$$\liminf_{n \rightarrow \infty} \|w_t(t)\|_{L^2(\Omega)}^2 \leq \|w_1\|_{L^2(\Omega)}^2,$$

which, together with $w_t \in C_s(0, 1; L^2(\Omega))$, yields (3.1). \square

Now, let us prove L^∞ regularity for the weak solutions. Decompose the weak solution determined by Lemma 3.1 as follows

$$w(t, x) = v(t, x) + u(t, x),$$

where

$$\begin{cases} v_{tt} - \Delta v_t + v_t - \Delta v = (1 + \lambda_1)w_t + \mu_1 w + h & \text{in } (0, 1) \times \Omega, \\ v = 0 & \text{on } (0, 1) \times \partial\Omega, \\ v(0, \cdot) = w_0, \quad v_t(0, \cdot) = w_1 & \text{in } \Omega \end{cases} \quad (3.15)$$

and

$$\begin{cases} u_{tt} - \Delta u_t + u_t - \Delta u = -f_1(w_t) - g_1(w) & \text{in } (0, 1) \times \Omega, \\ u = 0 & \text{on } (0, 1) \times \partial\Omega, \\ u(0, \cdot) = 0, \quad u_t(0, \cdot) = 0 & \text{in } \Omega. \end{cases} \quad (3.16)$$

Lemma 3.2. *Let $(w_0, w_1) \in (H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)$ and $w(t, x)$ be the weak solution of the problem (2.1). Then the problem (3.15) has a unique weak solution $v \in C([0, 1]; H_0^1(\Omega) \cap L^\infty(\Omega))$ with $v_t \in C([0, 1]; L^2(\Omega)) \cap L^2(0, 1; H_0^1(\Omega))$ such that*

$$\begin{aligned} & \|v(t) - e^{-t}w_0\|_{H^s(\Omega)} \leq \frac{c}{2-s} \left(1 + \|w\|_{C([0, 1]; L^2(\Omega))} \right. \\ & \left. + \|w_t\|_{L^\infty(0, 1; L^2(\Omega))} \right), \quad \forall t \in [0, 1], \quad \forall s \in [0, 2). \end{aligned} \quad (3.17)$$

Proof. Denoting $\varphi = v + v_t$, by (3.15), we have

$$\begin{cases} \varphi_t - \Delta \varphi = (1 + \lambda_1)w_t + \mu_1 w + h & \text{in } (0, 1) \times \Omega, \\ \varphi = 0 & \text{on } (0, 1) \times \partial\Omega, \\ \varphi(0, \cdot) = w_0 + w_1 & \text{in } \Omega. \end{cases} \quad (3.18)$$

It is well known (see, for example [23, p. 116]) that the Laplace operator Δ with $D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$ generates an analytic semigroup in $L^2(\Omega)$ and

$$\|e^{\Delta t}\|_{\mathcal{L}(L^2(\Omega), H^s(\Omega))} \leq Mt^{-\frac{s}{2}}, \quad (3.19)$$

for $t, s \geq 0$. Hence, by using the variations of constants formula, from (3.18) and (3.19), we obtain that $\varphi \in C([0, 1]; L^2(\Omega)) \cap C((0, 1]; H^s(\Omega)) \cap L^2(0, 1; H_0^1(\Omega))$ and

$$\|\varphi(t)\|_{H^s(\Omega)} \leq Mt^{-\frac{s}{2}} \|w_0 + w_1\|_{L^2(\Omega)} + \frac{2}{2-s} M(1 + \|w\|_{C([0,1]; L^2(\Omega))} + \|w_t\|_{L^\infty(0,1; L^2(\Omega))}), \quad (3.20)$$

for every $t \in [0, 1]$ and $s \in [0, 2)$. As a consequence, solving the equation $v + v_t = \varphi$, we get (3.17). By using the embedding $H^{1+\varepsilon}(\Omega) \subset L^\infty(\Omega)$, from (3.17), (3.20) and the equation $v + v_t = \varphi$, it follows that $v \in L^\infty(0, 1; H_0^1(\Omega) \cap L^\infty(\Omega))$ and $v_t \in L^1(0, 1; H_0^1(\Omega) \cap L^\infty(\Omega))$. Hence, $v \in C([0, 1]; H_0^1(\Omega) \cap L^\infty(\Omega))$, which, together with $\varphi \in C([0, 1]; L^2(\Omega)) \cap L^2(0, 1; H_0^1(\Omega))$, yields $v_t \in C([0, 1]; L^2(\Omega)) \cap L^2(0, 1; H_0^1(\Omega))$. \square

Lemma 3.3. *Assume that the conditions (2.2)-(2.4) are satisfied and $w(t, x)$ is the weak solution of the problem (2.1) on $[0, 1] \times \Omega$. Then $u \in C([0, 1]; L^\infty(\Omega))$ and for every $\delta > 0$ there exists $c(\delta) > 0$ such that*

$$\begin{aligned} & \|u(t)\|_{L^\infty(\Omega)} \leq \delta \\ & + c(\delta) \left(\int_0^1 \langle f_1(w_t(s)), w_t(s) \rangle ds + \int_0^1 \langle g_1(w_t(s)), w_t(s) \rangle ds \right), \quad \forall t \in [0, 1], \end{aligned} \quad (3.21)$$

where $u \in C([0, 1]; H_0^1(\Omega))$, with $u_t \in C_s(0, 1; L^2(\Omega)) \cap L^2(0, 1; H_0^1(\Omega))$ and $\lim_{t \searrow 0} \|u_t(t)\|_{L^2(\Omega)} = 0$, is the weak solution of (3.16).

Proof. Setting $h(t, x) = -f_1(w_t(t, x)) - g_1(w(t, x))$ and $v = u + u_t$, by (3.16), we have

$$\begin{cases} v_t - \Delta v = h(t, x) & \text{in } (0, 1) \times \Omega, \\ v = 0 & \text{on } (0, 1) \times \partial\Omega, \\ v(0, \cdot) = 0 & \text{in } \Omega. \end{cases}$$

By Lemma A.3, it follows that

$$\begin{aligned} |v(\tau, x)| & \leq \frac{1}{4\pi} \int_0^\tau \frac{1}{(\tau-s)} \int_\Omega e^{-\frac{|x-y|^2}{4(\tau-s)}} |h(s, y)| dy ds \\ & \leq \frac{1}{4\pi} \int_0^\tau \frac{1}{(\tau-s)} \int_\Omega e^{-\frac{|x-y|^2}{4(\tau-s)}} |g_1(w(s, y))| dy ds \\ & + \frac{1}{4\pi} \int_0^\tau \frac{1}{(\tau-s)} \int_\Omega e^{-\frac{|x-y|^2}{4(\tau-s)}} |f_1(w_t(s, y))| dy ds, \quad \text{a.e. in } (0, 1) \times \Omega. \end{aligned} \quad (3.22)$$

Now, let us estimate the right hand side of (3.22).

$$\begin{aligned} & \int_0^\tau \frac{1}{(\tau-s)} \int_\Omega e^{-\frac{|x-y|^2}{4(\tau-s)}} |g_1(w(s, y))| dy ds \\ & = \int_0^\tau \frac{1}{(\tau-s)} \int_{\{x \in \Omega: g_1^{-1}(-\varepsilon(\tau-s)^{-\alpha}) \leq w(s, y) \leq g_1^{-1}(\varepsilon(\tau-s)^{-\alpha})\}} e^{-\frac{|x-y|^2}{4(\tau-s)}} |g_1(w(s, y))| dy ds \\ & + \int_0^\tau \frac{1}{(\tau-s)} \int_{\{x \in \Omega: w(s, y) > g_1^{-1}(\varepsilon(\tau-s)^{-\alpha})\}} e^{-\frac{|x-y|^2}{4(\tau-s)}} |g_1(w(s, y))| dy ds \end{aligned}$$

$$\begin{aligned}
& + \int_0^\tau \frac{1}{\tau-s} \int_{\{x \in \Omega: w(s,y) < g_1^{-1}(-\varepsilon(\tau-s)^{-\alpha})\}} e^{-\frac{|x-y|^2}{4(\tau-s)}} |g_1(w(s,y))| dy ds \\
& \leq \varepsilon \int_0^\tau \frac{1}{(\tau-s)^{1+\alpha}} \int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4(\tau-s)}} dy ds + \int_0^\tau \frac{1}{(\tau-s) g_1^{-1}(\varepsilon(\tau-s)^{-\alpha})} \int_\Omega g_1(w(s,y)) w(s,y) dy ds \\
& \quad - \int_0^\tau \frac{1}{(\tau-s) g_1^{-1}(-\varepsilon(\tau-s)^{-\alpha})} \int_\Omega g_1(w(s,y)) w(s,y) dy ds \\
& = 4\pi\varepsilon \int_0^\tau \frac{1}{(\tau-s)^\alpha} ds + \int_0^\tau \frac{1}{(\tau-s) g_1^{-1}(\varepsilon(\tau-s)^{-\alpha})} \int_\Omega g_1(w(s,y)) w(s,y) dy ds \\
& \quad - \int_0^\tau \frac{1}{(\tau-s) g_1^{-1}(-\varepsilon(\tau-s)^{-\alpha})} \int_\Omega g_1(w(s,y)) w(s,y) dy ds, \quad \forall (\tau, x) \in (0, 1) \times \Omega,
\end{aligned}$$

where $\alpha \in (0, 1)$. Hence, we have

$$\begin{aligned}
& \int_0^t \left\| \int_0^\tau \frac{1}{(\tau-s)} \int_\Omega e^{-\frac{|x-y|^2}{4(\tau-s)}} |g_1(w(s,y))| dy ds \right\|_{L^\infty(\Omega)} d\tau \\
& \leq \frac{4\pi\varepsilon}{(2-\alpha)(1-\alpha)} t^{2-\alpha} \\
& + \int_0^t \int_0^\tau \left(\frac{1}{(\tau-s) g_1^{-1}(\varepsilon(\tau-s)^{-\alpha})} - \frac{1}{(\tau-s) g_1^{-1}(-\varepsilon(\tau-s)^{-\alpha})} \right) \int_\Omega g_1(w(s,y)) w(s,y) dy ds d\tau \\
& = \frac{4\pi\varepsilon}{(2-\alpha)(1-\alpha)} t^{2-\alpha} \\
& + \int_0^t \int_\Omega g_1(w(s,y)) w(s,y) dy \int_s^t \left(\frac{1}{(\tau-s) g_1^{-1}(\varepsilon(\tau-s)^{-\alpha})} - \frac{1}{(\tau-s) g_1^{-1}(-\varepsilon(\tau-s)^{-\alpha})} \right) d\tau ds, \quad (3.23)
\end{aligned}$$

By the condition (2.4), we obtain

$$\begin{aligned}
& \int_s^t \left(\frac{1}{(t-s) g_1^{-1}(\varepsilon(t-s)^{-\alpha})} - \frac{1}{(t-s) g_1^{-1}(-\varepsilon(t-s)^{-\alpha})} \right) d\tau \\
& = \int_0^{t-s} \left(\frac{1}{\sigma g_1^{-1}(\varepsilon\sigma^{-\alpha})} - \frac{1}{\sigma g_1^{-1}(-\varepsilon\sigma^{-\alpha})} \right) d\sigma \leq \frac{1}{\alpha} \int_1^\infty \frac{1}{\lambda g_1^{-1}(\varepsilon\lambda)} d\lambda \\
& \quad - \frac{1}{\alpha} \int_1^\infty \frac{1}{\lambda g_1^{-1}(-\varepsilon\lambda)} d\lambda = \frac{1}{\alpha} \int_\varepsilon^\infty \frac{1}{\lambda g_1^{-1}(\lambda)} d\lambda \\
& \quad - \frac{1}{\alpha} \int_\varepsilon^\infty \frac{1}{\lambda g_1^{-1}(-\lambda)} d\lambda = \frac{1}{\alpha} \int_{g_1^{-1}(\varepsilon)}^\infty \frac{g_1'(\nu)}{\nu g_1(\nu)} d\nu \\
& \quad + \frac{1}{\alpha} \int_{-\infty}^{g_1^{-1}(-\varepsilon)} \frac{g_1'(\nu)}{\nu g_1(\nu)} d\nu < \infty,
\end{aligned}$$

which, together with (3.23), yields

$$\begin{aligned} \int_0^t \left\| \int_0^\tau \frac{1}{(\tau-s)} \int_\Omega e^{-\frac{|x-y|^2}{4(\tau-s)}} |g_1(w(s, y))| dy ds \right\|_{L^\infty(\Omega)} d\tau &\leq \frac{4\pi\varepsilon}{(2-\alpha)(1-\alpha)} \\ &+ k_\varepsilon \int_0^1 \int_\Omega g_1(w(s, y)) w(s, y) dy ds, \quad \forall t \in [0, 1], \end{aligned}$$

where $k_\varepsilon = \frac{1}{\alpha} \left(\int_{g_1^{-1}(\varepsilon)}^\infty \frac{g'_1(\nu)}{\nu g_1(\nu)} d\nu + \int_{-\infty}^{g_1^{-1}(-\varepsilon)} \frac{g'_1(\nu)}{\nu g_1(\nu)} d\nu \right)$. Similarly, we find

$$\begin{aligned} \int_0^t \left\| \int_0^\tau \frac{1}{(\tau-s)} \int_\Omega e^{-\frac{|x-y|^2}{4(\tau-s)}} |f_1(w_t(s, y))| dy ds \right\|_{L^\infty(\Omega)} d\tau &\leq \frac{4\pi\varepsilon}{(2-\alpha)(1-\alpha)} \\ &+ \tilde{k}_\varepsilon \int_0^1 \int_\Omega f_1(w_t(s, y)) w_t(s, y) dy ds, \quad \forall t \in [0, 1], \end{aligned}$$

where $\tilde{k}_\varepsilon = \frac{1}{\alpha} \left(\int_{f_1^{-1}(\varepsilon)}^\infty \frac{f'_1(\nu)}{\nu f_1(\nu)} d\nu + \int_{-\infty}^{f_1^{-1}(-\varepsilon)} \frac{f'_1(\nu)}{\nu f_1(\nu)} d\nu \right)$ and $\alpha \in (0, 1)$. Taking into account the last two inequalities in (3.22), we get $v \in L^1(0, 1; L^\infty(\Omega))$ and

$$\begin{aligned} \int_0^1 \|v(\tau)\|_{L^\infty(\Omega)} d\tau &\leq \frac{2\varepsilon}{(2-\alpha)(1-\alpha)} + \frac{\tilde{k}_\varepsilon}{4\pi} \int_0^1 \int_\Omega f_1(w_t(s, y)) w_t(s, y) dy ds \\ &+ \frac{k_\varepsilon}{4\pi} \int_0^1 \int_\Omega g_1(w(s, y)) w(s, y) dy ds, \quad \forall \varepsilon > 0. \end{aligned} \quad (3.24)$$

Now, solving the problem

$$\begin{cases} u_t(t, x) + u(t, x) = v(t, x) & \text{in } (0, 1) \times \Omega, \\ u(0, x) = 0 & \text{in } \Omega \end{cases}$$

and taking into account (3.24), we obtain $u \in C([0, 1]; L^\infty(\Omega))$, $u_t \in L^1(0, 1; L^\infty(\Omega))$ and

$$\begin{aligned} \|u(t)\|_{L^\infty(\Omega)} &\leq \int_0^t e^{-(t-\tau)} \|v(\tau)\|_{L^\infty(\Omega)} d\tau \leq \int_0^t \|v(\tau)\|_{L^\infty(\Omega)} d\tau \\ &\leq \frac{2\varepsilon}{(2-\alpha)(1-\alpha)} + \frac{\tilde{k}_\varepsilon}{4\pi} \int_0^1 \langle f_1(w_t(s)), w_t(s) \rangle ds \\ &+ \frac{k_\varepsilon}{4\pi} \int_0^1 \langle g_1(w_t(s)), w_t(s) \rangle ds, \quad \forall t \in [0, 1] \text{ and } \forall \varepsilon > 0, \end{aligned} \quad (3.25)$$

which yields (3.21). \square

Thus, by Lemma 3.1-3.3, it follows that the problem (2.1) has a weak solution $w \in C([0, 1]; H_0^1(\Omega) \cap L^\infty(\Omega))$ with $w_t \in C_s(0, 1; L^2(\Omega)) \cap L^2(0, 1; H_0^1(\Omega))$. Also, by (3.7), (3.10), (3.14), (3.17) and (3.21), we have

$$\|w(t)\|_{L^\infty(\Omega)} \leq c(\|(w_0, w_1)\|_{(H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)}), \quad \forall t \in [0, 1], \quad (3.26)$$

where $c : R_+ \rightarrow R_+$ is a nondecreasing function.

Now, we are in a position to prove the uniqueness of the weak solution.

Lemma 3.4. *Under the conditions (2.2)-(2.4), the problem (2.1) has a unique weak solution.*

Proof. Assume that there are two solutions to (2.1) as $w^{(i)} \in C([0, 1]; H_0^1(\Omega) \cap L^\infty(\Omega))$ with $w_t^{(i)} \in C_s(0, 1; L^2(\Omega)) \cap L^2(0, 1; H_0^1(\Omega))$, $w^{(i)}(0) = w_0$ and $\lim_{t \searrow 0} \|w_t^{(i)}(t) - w_1\|_{L^2(\Omega)} = 0$, $i = 1, 2$. Let $u(t, x) = w_1(t, x) - w_2(t, x)$. Testing the equation

$$u_{tt} - \Delta u_t + f(w_t^{(1)}) - f(w_t^{(2)}) - \Delta u + g(w^{(1)}) - g(w^{(2)}) = 0,$$

by $\frac{u(t)-u(t-h)}{2h}$ and $\frac{u(t+h)-u(t)}{2h}$ on $(\tau+h, s+h) \times \Omega$ and $(\tau, s) \times \Omega$, respectively, and then summing these relations, we obtain

$$\begin{aligned} & \frac{1}{2h} \langle u_t(s+h) + u_t(s), u(s+h) - u(s) \rangle - \frac{1}{2h} \langle u_t(\tau+h) + u_t(\tau), u(\tau+h) - u(\tau) \rangle \\ & - \frac{1}{2h} \int_{\tau}^s \langle u_t(t+h) + u_t(t), u_t(t+h) - u_t(t) \rangle dt \\ & + \frac{1}{2h} \int_{\tau}^s \langle \nabla u_t(t+h) + \nabla u_t(t), \nabla u(t+h) - \nabla u(t) \rangle dt \\ & + \int_{\tau}^s \left\langle f(w_t^{(1)}(t)) - f(w_t^{(2)}(t)), \frac{u(t+h) - u(t-h)}{2h} \right\rangle dt \\ & + \int_s^{s+h} \left\langle f(w_t^{(1)}(t)) - f(w_t^{(2)}(t)), \frac{u(t) - u(t-h)}{2h} \right\rangle dt \\ & - \int_{\tau}^{\tau+h} \left\langle f(w_t^{(1)}(t)) - f(w_t^{(2)}(t)), \frac{u(t+h) - u(t)}{2h} \right\rangle dt \\ & + \frac{1}{2h} \int_{\tau}^s \langle \nabla u(t+h) + \nabla u(t), \nabla u(t+h) - \nabla u(t) \rangle dt \\ & = \frac{1}{2} \int_{\tau}^s \left\langle g(w^{(2)}(t+h)) - g(w^{(1)}(t+h)), \frac{u(t+h) - u(t)}{h} \right\rangle dt \\ & + \frac{1}{2} \int_{\tau}^s \left\langle g(w^{(2)}(t)) - g(w^{(1)}(t)), \frac{u(t+h) - u(t)}{h} \right\rangle dt, \quad \forall [\tau, s] \subset (0, 1), \end{aligned} \quad (3.27)$$

where h is a sufficiently small positive number. Since

$$\begin{aligned} & \frac{1}{2h} \langle u_t(s+h) + u_t(s), u(s+h) - u(s) \rangle - \frac{1}{2h} \int_{\tau}^s \langle u_t(t+h) + u_t(t), u_t(t+h) - u_t(t) \rangle dt \\ & + \frac{1}{2h} \int_{\tau}^s \langle \nabla u(t+h) + \nabla u(t), \nabla u(t+h) - \nabla u(t) \rangle dt = \frac{1}{4h} \frac{d}{ds} \langle u(s+h) - u(s), u(s+h) - u(s) \rangle \\ & + \frac{1}{h} \langle u_t(s), u(s+h) - u(s) \rangle - \frac{1}{2h} \int_s^{s+h} \|u_t(t)\|_{L^2(\Omega)}^2 dt + \frac{1}{2h} \int_{\tau}^{\tau+h} \|u_t(t)\|_{L^2(\Omega)}^2 dt \\ & + \frac{1}{2h} \int_s^{s+h} \|\nabla u(t)\|_{L^2(\Omega)}^2 dt - \frac{1}{2h} \int_{\tau}^{\tau+h} \|\nabla u(t)\|_{L^2(\Omega)}^2 dt \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2} \int_{\tau}^s \left\langle g(w^{(2)}(t+h)) - g(w^{(1)}(t+h)), \frac{u(t+h) - u(t)}{h} \right\rangle dt \\ & + \frac{1}{2} \int_{\tau}^s \left\langle g(w^{(2)}(t)) - g(w^{(1)}(t)), \frac{u(t+h) - u(t)}{h} \right\rangle dt \\ & \leq c \int_{\tau}^s \left(\|u(t+h)\|_{L^2(\Omega)} + \|u(t)\|_{L^2(\Omega)} \right) \left\| \frac{u(t+h) - u(t)}{h} \right\|_{L^2(\Omega)} dt, \end{aligned}$$

integrating (3.27) on (τ, T) with respect to s , passing to the limit as $h \searrow 0$ and taking into account Lemma A.4-A.5, we get

$$\begin{aligned} & \int_{\tau}^T \|u_t(s)\|_{L^2(\Omega)}^2 ds + \frac{1}{2} \int_{\tau}^T \|\nabla u(s)\|_{L^2(\Omega)}^2 ds + \int_{\tau}^T \|\nabla u_t(s)\|_{L^2(\Omega)}^2 ds \\ & \leq \frac{1}{2} \limsup_{h \searrow 0} \frac{1}{h} \int_{\tau}^T \int_s^{s+h} \|u_t(t)\|_{L^2(\Omega)}^2 dt ds + (T - \tau) \|\nabla u(\tau)\|_{L^2(\Omega)}^2 + \widehat{c}(T - \tau) \|u_t(\tau)\|_{L^2(\Omega)} \\ & \quad + 2c \int_0^T \int_0^s \|u(t)\|_{L^2(\Omega)} \|u_t(t)\|_{L^2(\Omega)} dt ds, \quad \forall [\tau, T] \subset (0, 1). \end{aligned} \quad (3.28)$$

By Lebesgue's convergence theorem, we find

$$\begin{aligned} \lim_{h \searrow 0} \frac{1}{h} \int_{\tau}^T \int_s^{s+h} \|u_t(t)\|_{L^2(\Omega)}^2 dt ds &= \lim_{h \searrow 0} \frac{1}{h} \int_{\tau}^T \int_0^h \|u_t(s+t)\|_{L^2(\Omega)}^2 dt ds = \lim_{h \searrow 0} \int_{\tau}^T \int_0^1 \|u_t(s+h\sigma)\|_{L^2(\Omega)}^2 d\sigma ds \\ &= \lim_{h \searrow 0} \int_0^1 \int_{\tau}^T \|u_t(s+h\sigma)\|_{L^2(\Omega)}^2 ds d\sigma = \lim_{h \searrow 0} \int_0^1 \int_{\tau+h\sigma}^{T+h\sigma} \|u_t(\tau)\|_{L^2(\Omega)}^2 d\tau d\sigma = \int_{\tau}^T \|u_t(\tau)\|_{L^2(\Omega)}^2 d\tau. \end{aligned}$$

Hence, taking the last equality into consideration in (3.28) and then passing to the limit as $\tau \searrow 0$, we have

$$\int_0^T \left(\|u_t(s)\|_{L^2(\Omega)}^2 + \|\nabla u(s)\|_{L^2(\Omega)}^2 \right) ds \leq 2c \int_0^T \int_0^s \left(\|u(t)\|_{L^2(\Omega)}^2 + \|u_t(t)\|_{L^2(\Omega)}^2 \right) dt ds, \quad \forall T \in [0, 1].$$

Denoting $y(s) = \int_0^s \left(\|u(t)\|_{L^2(\Omega)}^2 + \|u_t(t)\|_{L^2(\Omega)}^2 \right) dt$, by the last inequality, it follows that

$$y(T) \leq 2c \int_0^T y(s) ds, \quad \forall T \in [0, 1].$$

Thus, applying Gronwall's lemma, we obtain $y(s) = 0$ and consequently $u(s, \cdot) = 0$, for every $s \in [0, 1]$. \square

Now, putting $(w(1), w_t(1))$ instead of (w_0, w_1) in (2.1), by Lemma 3.1-3.3, we see that the problem has a unique weak solution $v(t, x)$ on $[0, 1] \times \Omega$ with $v \in C([0, 1]; L^\infty(\Omega))$. Also, it is easy to see that the function $\tilde{w}(t, x) = \begin{cases} w(t, x), & \forall (t, x) \in [0, 1] \times \Omega \\ v(t-1, x), & \forall (t, x) \in (1, 2] \times \Omega \end{cases}$ is a weak solution to (2.1) on $[0, 2] \times \Omega$ with $\tilde{w} \in C([0, 2]; L^\infty(\Omega))$. Hence, continuing this procedure we can extend the weak solution of (2.1) to $[0, T] \times \Omega$, for every $T > 0$. Furthermore, the inequalities (3.14) and (3.26) also can be extended to $[0, T]$, which give us (2.5).

Denoting $H(t, x) = -g(w(t, x) + h(x))$, by (3.26), we have $H \in L^\infty(0, T; L^2(\Omega))$. Also, due to (2.1),

$$\begin{cases} w_{tt} - \Delta w_t + f(w_t) - \Delta w = H(t, x) & \text{in } (0, T) \times \Omega, \\ w = 0 & \text{on } (0, T) \times \partial\Omega, \\ w(0, \cdot) = w_0, \quad w_t(0, \cdot) = w_1 & \text{in } \Omega. \end{cases} \quad (3.29)$$

Applying the techniques of the proof of Lemma 3.3, we can say that the function $w \in C([0, T]; H_0^1(\Omega) \cap L^\infty(\Omega))$ with $w_t \in C_s(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$, is a unique solution to (3.29).

Formally multiplying (3.29) by w_{tt} and integrating over $(s, T) \times \Omega$, we obtain

$$\begin{aligned} & \frac{1}{2} \int_s^T \|w_{tt}(t)\|_{L^2(\Omega)}^2 dt + \frac{1}{2} \|\nabla w_t(T)\|_{L^2(\Omega)}^2 + \langle F(w(T)), 1 \rangle + \langle \nabla w(T), \nabla w_t(T) \rangle \\ & \leq \int_s^T \|\nabla w_t(t)\|_{L^2(\Omega)}^2 dt + \frac{1}{2} \|\nabla w_t(s)\|_{L^2(\Omega)}^2 + \langle F(w(s)), 1 \rangle \\ & \quad + \langle \nabla w(s), \nabla w_t(s) \rangle + \frac{1}{2} \int_s^T \|H(t)\|_{L^2(\Omega)}^2 dt, \end{aligned}$$

where $F(w) = \int_0^w f(s) ds$. Integrating the last equality on $[0, T]$ with respect to s and taking into account (2.2) and (2.5), we find

$$\int_0^T \int_s^T \|w_{tt}(t)\|_{L^2(\Omega)}^2 dt ds \leq \tilde{c}(T, \|(w_0, w_1)\|_{(H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)}). \quad (3.30)$$

Since the problem (3.29) admits a unique solution, using Galerkin's approximations one can justify (3.30). So, by (3.30), we have $w_{tt} \in L_{loc}^2(0, T; L^2(\Omega))$, which together with $w_t \in C_s(0, T; L^2(\Omega))$ and (3.1) implies that $w_t \in C([0, T]; L^2(\Omega))$.

Thus, to finish the proof of Theorem 2.1, we just need to show (2.6). Let $w, v \in C([0, T]; H_0^1(\Omega) \cap L^\infty(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \cap W^{1,2}(0, T; H_0^1(\Omega)) \cap W_{loc}^{2,2}(0, T; L^2(\Omega))$ are the weak solutions to (2.1). Then due to (2.1)₁, the function $u(t, x) = w(t, x) - v(t, x)$ satisfies the equation

$$u_{tt} - \Delta u_t + f(w_t) - f(v_t) - \Delta u = g(v) - g(w).$$

Testing the above equation by $2u_t$ on $(s, t) \times \Omega$ and taking into account (2.2) and (2.5), we get

$$\begin{aligned} \|u_t(t)\|_{L^2(\Omega)}^2 + \|\nabla u(t)\|_{L^2(\Omega)}^2 & \leq M \int_s^t \left(\|u_t(\tau)\|_{L^2(\Omega)}^2 + \|\nabla u(\tau)\|_{L^2(\Omega)}^2 \right) d\tau \\ & \quad + \|u_t(s)\|_{L^2(\Omega)}^2 + \|\nabla u(s)\|_{L^2(\Omega)}^2, \quad 0 < s \leq t < T. \end{aligned}$$

Therefore, applying Gronwall's lemma, we obtain (2.6).

4. DISSIPATIVITY AND ASYMPTOTIC COMPACTNESS

We begin with the following dissipativity result.

Lemma 4.1. *Assume that the conditions (2.2)-(2.4) are satisfied. Then for every bounded subset $B \subset (H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)$, there exists $c_B > 0$ such that*

$$\sup_{\varphi \in B} \|S(t)\varphi\|_{(H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)} \leq c_B, \quad \forall t \geq 0. \quad (4.1)$$

Proof. Let $\varphi \in B$ and $(w(t), w_t(t)) = S(t)\varphi$. By (2.5), immediately, it follows that

$$\|S(t)\varphi\|_{H_0^1(\Omega) \times L^2(\Omega)} \leq c_B^{(1)}, \quad \forall t \geq 0. \quad (4.2)$$

Denote by $v^{(s)}(t, x)$ the weak solution of (3.15) with $(w(s), w_t(s))$ and $w(t+s)$ instead of (w_0, w_1) and $w(t)$, respectively. Also, denote $u^{(s)}(t, x) = w(t+s, x) - v^{(s)}(t, x)$, for $(t, x) \in [0, 1] \times \Omega$ and $s \geq 0$. Then by Lemma 3.2-3.3 and (2.5), we find

$$\begin{aligned} \|w(t+s)\|_{L^\infty(\Omega)} &\leq e^{-t} \|w(s)\|_{L^\infty(\Omega)} \\ &+ c \int_s^{t+s} \langle g_1(w(\tau)), w(\tau) \rangle d\tau + c_B^{(2)}, \quad \forall t \in [0, 1] \text{ and } \forall s \geq 0. \end{aligned} \quad (4.3)$$

Testing (2.1) by $w(t, x)$ on $(s, t+s) \times \Omega$ and taking into account (2.5), we get

$$\begin{aligned} \int_s^{t+s} \langle g_1(w(\tau)), w(\tau) \rangle d\tau &\leq - \int_s^{t+s} \langle f_1(w_t(\tau)), w(\tau) \rangle d\tau + c_B^{(3)} \\ &= - \int_0^t \langle f_1(w_t(\tau+s)), w(\tau+s) \rangle d\tau + c_B^{(3)}, \quad \forall t \in [0, 1] \text{ and } \forall s \geq 0. \end{aligned}$$

Applying Lemma A.6 to the integral on the right hand side of the last equality, gives us

$$\begin{aligned} \int_s^{t+s} \langle g_1(w(\tau)), w(\tau) \rangle d\tau &\leq \frac{1}{M} \|w(s)\|_{L^\infty(\Omega)} \int_0^t \langle f_1(w_t(\tau+s)), w_t(\tau+s) \rangle d\tau \\ &+ \int_0^t \langle f_1(w_t(\tau+s)), w_t(\tau+s) \rangle d\tau + \|f_1\|_{C[-M, M]} \|w(s)\|_{L^1(\Omega)} \\ &+ \int_0^t |\langle f_1(-w_t(\tau+s)), w_t(\tau+s) \rangle| d\tau, \end{aligned} \quad (4.4)$$

for every $M > 0$, $t \in [0, 1]$ and $s \geq 0$. So, by (2.3), (2.5), (4.3) and (4.4), we have

$$\|w(t+s)\|_{L^\infty(\Omega)} \leq (e^{-t} + \frac{c_B^{(4)}}{M}) \|w(s)\|_{L^\infty(\Omega)} + c_B^{(4)} (1 + \|f_1\|_{C[-M, M]}), \quad (4.5)$$

for every $M > 0$, $t \in [0, 1]$ and $s \geq 0$. Choosing $t = 1$ and $M = ec_B^{(4)}$ in (4.5), we obtain

$$\|w(s+1)\|_{L^\infty(\Omega)} \leq \frac{2}{e} \|w(s)\|_{L^\infty(\Omega)} + c_B^{(5)}, \quad \forall s \geq 0.$$

Thus, by the last inequality, it follows that

$$\|w(n)\|_{L^\infty(\Omega)} \leq \left(\frac{2}{e}\right)^n \|w_0\|_{L^\infty(\Omega)} + c_B^{(5)} \frac{1 - \left(\frac{2}{e}\right)^{n-1}}{1 - \frac{2}{e}}$$

and consequently

$$\|w(n)\|_{L^\infty(\Omega)} \leq \|w_0\|_{L^\infty(\Omega)} + \frac{e}{e-2} c_B^{(5)}, \quad \forall n \in \mathbb{Z}_+. \quad (4.6)$$

Since for every $T \geq 0$ there exist $n_T \in \mathbb{Z}_+$ and $t_T \in [0, 1]$ such that

$$T = n_T + t_T,$$

by (4.5) and (4.6), we get

$$\|w(T)\|_{L^\infty(\Omega)} \leq c_B^{(6)}, \quad \forall T \geq 0,$$

which, together with (4.2), yields (4.1). \square

Now, we will show asymptotic compactness of $\{S(t)\}_{t \geq 0}^\infty$ in $(H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)$.

Lemma 4.2. *Let the conditions (2.2)-(2.4) be satisfied and B be bounded subset of $(H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)$. Then the set $\{S(t_m)\varphi_m\}_{m=1}^\infty$ is relatively compact in $(H_0^1(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)$, where $t_m \rightarrow \infty$ and $\{\varphi_m\}_{m=1}^\infty \subset B$.*

Proof. Step1. We first prove relatively compactness of $\{S(t_m)\varphi_m\}_{m=1}^\infty$ in $H_0^1(\Omega) \times L^2(\Omega)$. For any $T > 0$ and $m \in \mathbb{N}$ such that $t_m \geq T$, let us define

$$(w_m^{(T)}(t), w_{mt}^{(T)}(t)) = S(t + t_m - T)\varphi_m.$$

By Lemma 4.1 and (2.5)₁, it follows that

$$\begin{aligned} & \|w_m^{(T)}(t)\|_{H_0^1(\Omega) \cap L^\infty(\Omega)} + \|w_{mt}^{(T)}(t)\|_{L^2(\Omega)} \\ & + \int_0^t \|\nabla w_{mt}^{(T)}(s)\|_{L^2(\Omega)}^2 ds + \int_0^t \langle f_1(w_{mt}^{(T)}(s)), w_{mt}^{(T)}(s) \rangle ds \leq c_1, \quad \forall t \geq 0. \end{aligned} \quad (4.7)$$

Then, by Banach-Alaoglu theorem, there exist a subsequence $\{w_{m_n}^{(T)}(t)\}_{n=1}^\infty$ and a function $w \in L^\infty(0, T; H_0^1(\Omega) \cap L^\infty(\Omega)) \cap W^{1,\infty}(0, T; L^2(\Omega)) \cap W^{1,2}(0, T; H_0^1(\Omega))$ such that

$$\begin{cases} w_{m_n}^{(T)}(t) \rightarrow w(t) \text{ weakly star in } L^\infty(0, T; H_0^1(\Omega) \cap L^\infty(\Omega)), \\ w_{m_n t}^{(T)}(t) \rightarrow w_t(t) \text{ weakly star in } L^\infty(0, T; L^2(\Omega)), \\ w_{m_n t}^{(T)}(t) \rightarrow w_t(t) \text{ weakly in } L^2(0, T; H_0^1(\Omega)). \end{cases} \quad (4.8)$$

Testing the equation

$$\begin{aligned} & w_{m_n t t}^{(T)} - w_{m_k t t}^{(T)} - \Delta(w_{m_n t}^{(T)} - w_{m_k t}^{(T)}) - \Delta(w_{m_n}^{(T)} - w_{m_k}^{(T)}) + f(w_{m_n t}^{(T)}) - f(w_{m_k t}^{(T)}) \\ & + g(w_{m_n}^{(T)}) - g(w_{m_k}^{(T)}) = 0 \end{aligned} \quad (4.9)$$

by $2(w_{m_n t}^{(T)} - w_{m_k t}^{(T)})$ on $(s, T) \times \Omega$ and taking into account (2.2) and (4.7), we find

$$\begin{aligned} & \|w_{m_n t}^{(T)}(T) - w_{m_k t}^{(T)}(T)\|_{L^2(\Omega)}^2 + \|\nabla(w_{m_n}^{(T)}(T) - w_{m_k}^{(T)}(T))\|_{L^2(\Omega)}^2 \\ & + c_1 \int_s^T \|w_{m_n t}^{(T)}(t) - w_{m_k t}^{(T)}(t)\|_{L^2(\Omega)}^2 dt \leq \|w_{m_n t}^{(T)}(s) - w_{m_k t}^{(T)}(s)\|_{L^2(\Omega)}^2 + \|\nabla(w_{m_n}^{(T)}(s) - w_{m_k}^{(T)}(s))\|_{L^2(\Omega)}^2 \\ & + c_2 \int_s^T \|w_{m_n}^{(T)}(t) - w_{m_k}^{(T)}(t)\|_{L^2(\Omega)}^2 dt. \end{aligned} \quad (4.10)$$

Also, testing (4.9) by $(w_{m_n}^{(T)} - w_{m_k}^{(T)})$ on $(0, T) \times \Omega$ and considering (4.7), we get

$$\begin{aligned} & \int_0^T \|\nabla(w_{m_n}^{(T)}(t) - w_{m_k}^{(T)}(t))\|_{L^2(\Omega)}^2 dt \leq c_3 + \int_0^T \|w_{m_n t}^{(T)}(t) - w_{m_k t}^{(T)}(t)\|_{L^2(\Omega)}^2 dt \\ & + c_4 \int_0^T \|w_{m_n}^{(T)}(t) - w_{m_k}^{(T)}(t)\|_{L^2(\Omega)}^2 dt \\ & + \left| \int_0^T \langle f(w_{m_n t}^{(T)}(t)) - f(w_{m_k t}^{(T)}(t)), w_{m_n}^{(T)}(t) - w_{m_k}^{(T)}(t) \rangle dt \right|, \end{aligned}$$

which, together with (4.10), implies that

$$\int_0^T \|w_{m_n t}^{(T)}(t) - w_{m_k t}^{(T)}(t)\|_{L^2(\Omega)}^2 dt + \int_0^T \|\nabla(w_{m_n}^{(T)}(t) - w_{m_k}^{(T)}(t))\|_{L^2(\Omega)}^2 dt$$

$$\begin{aligned}
&\leq c_5 + c_6 \int_0^T \left\| w_{m_n}^{(T)}(t) - w_{m_k}^{(T)}(t) \right\|_{L^2(\Omega)}^2 dt + \\
&+ \left| \int_0^T \left\langle f(w_{m_n}^{(T)}(t)) - f(w_{m_k}^{(T)}(t)), w_{m_n}^{(T)}(t) - w_{m_k}^{(T)}(t) \right\rangle dt \right|. \tag{4.11}
\end{aligned}$$

Integrating (4.10) over $(0, T)$ with respect to s and taking (4.11) into consideration, we obtain

$$\begin{aligned}
&T \left\| w_{m_n}^{(T)}(T) - w_{m_k}^{(T)}(T) \right\|_{L^2(\Omega)}^2 + T \left\| \nabla(w_{m_n}^{(T)}(T) - w_{m_k}^{(T)}(T)) \right\|_{L^2(\Omega)}^2 \\
&\leq c_5 + c_7(1 + T) \int_0^T \left\| w_{m_n}^{(T)}(t) - w_{m_k}^{(T)}(t) \right\|_{L^2(\Omega)}^2 dt \\
&+ \left| \int_0^T \left\langle f(w_{m_n}^{(T)}(t)) - f(w_{m_k}^{(T)}(t)), w_{m_n}^{(T)}(t) - w_{m_k}^{(T)}(t) \right\rangle dt \right|. \tag{4.12}
\end{aligned}$$

By using the compact embedding theorem (see [21]), from (4.8), it follows that

$$\lim_{n, k \rightarrow \infty} \int_0^T \left\| w_{m_n}^{(T)}(t) - w_{m_k}^{(T)}(t) \right\|_{L^2(\Omega)}^2 dt = 0$$

and taking into account (4.7), we have

$$\begin{aligned}
&\limsup_{n \rightarrow \infty} \limsup_{k \rightarrow \infty} \left| \int_0^T \left\langle f(w_{m_n}^{(T)}(t)) - f(w_{m_k}^{(T)}(t)), w_{m_n}^{(T)}(t) - w_{m_k}^{(T)}(t) \right\rangle dt \right| \\
&\leq \limsup_{n \rightarrow \infty} \limsup_{k \rightarrow \infty} \left| \int_0^T \int_{\{x: x \in \Omega \mid |w_{m_n}^{(T)}(t, x)| > 1\}} \left\langle f(w_{m_n}^{(T)}(t)) - f(w_{m_k}^{(T)}(t)), w_{m_n}^{(T)}(t) - w_{m_k}^{(T)}(t) \right\rangle dt \right| \\
&\leq c_8 \limsup_{n \rightarrow \infty} \int_0^T \left\langle f_1(w_{m_n}^{(T)}(t)), w_{m_n}^{(T)}(t) \right\rangle dt + c_8 \limsup_{n \rightarrow \infty} \int_0^T \left\| w_{m_n}^{(T)}(t) \right\|_{L^2(\Omega)}^2 dt \leq c_9.
\end{aligned}$$

Hence, passing to the limit in (4.12), we obtain

$$\limsup_{n \rightarrow \infty} \limsup_{k \rightarrow \infty} \|S(t_{m_n})\varphi_{m_n} - S(t_{m_k})\varphi_{m_k}\|_{H_0^1(\Omega) \times L^2(\Omega)} \leq \frac{c_{10}}{\sqrt{T}}, \quad \forall T > 0$$

and consequently

$$\lim_{n \rightarrow \infty} \limsup_{k \rightarrow \infty} \|S(t_{m_n})\varphi_{m_n} - S(t_{m_k})\varphi_{m_k}\|_{H_0^1(\Omega) \times L^2(\Omega)} = 0. \tag{4.13}$$

So, passing to the limit in the inequality

$$\begin{aligned}
&\|S(t_{m_n})\varphi_{m_n} - S(t_{m_\nu})\varphi_{m_\nu}\|_{H_0^1(\Omega) \times L^2(\Omega)} \\
&\leq \limsup_{k \rightarrow \infty} \|S(t_{m_n})\varphi_{m_n} - S(t_{m_k})\varphi_{m_k}\|_{H_0^1(\Omega) \times L^2(\Omega)} + \limsup_{k \rightarrow \infty} \|S(t_{m_k})\varphi_{m_k} - S(t_{m_\nu})\varphi_{m_\nu}\|_{H_0^1(\Omega) \times L^2(\Omega)}
\end{aligned}$$

and taking (4.13) into consideration, we get

$$\lim_{n, \nu \rightarrow \infty} \|S(t_{m_n})\varphi_{m_n} - S(t_{m_\nu})\varphi_{m_\nu}\|_{H_0^1(\Omega) \times L^2(\Omega)} = 0.$$

Thus, the subsequence $\{S(t_{m_n})\varphi_{m_n}\}_{n=1}^\infty$ is a Cauchy sequence in $H_0^1(\Omega) \times L^2(\Omega)$ and consequently converges. By the same way, we can show that every subsequence of $\{S(t_m)\varphi_m\}_{m=1}^\infty$ has a convergent subsequence in $H_0^1(\Omega) \times L^2(\Omega)$. This gives us relatively compactness of $\{S(t_m)\varphi_m\}_{m=1}^\infty$ in $H_0^1(\Omega) \times L^2(\Omega)$.

Step2. Now, let us prove relatively compactness of $\{w_m(t_m)\}_{m=1}^\infty$ in $L^\infty(\Omega)$, where $(w_m(t), w_{mt}(t) = S(t)\varphi_m)$. By the relatively compactness of $\{S(t_m)\varphi_m\}_{m=1}^\infty$ in $H_0^1(\Omega) \times L^2(\Omega)$, for any $\delta > 0$ there exist $T_\delta > 0$ and $M_\delta > 0$ such that

$$\int_{\{x: x \in \Omega \mid |w_{mt}(t, x)| > M_\delta\}} (|\nabla w_m(t, x)|^2 + |w_{mt}(t, x)|^2) dx < \delta, \quad \forall t \geq T_\delta. \quad (4.13)$$

Testing the equation

$$w_{mtt} - \Delta w_{mt} + f(w_{mt}) - \Delta w_m + g(w_m) = h$$

by $\begin{cases} w_{mt} + M_\delta, & w_{mt} \leq -M_\delta, \\ 0, & |w_{mt}| \leq M_\delta, \\ w_{mt} - M_\delta, & w_{mt} \geq M_\delta \end{cases}$ on $(s, s+1) \times \Omega$ and taking into account (4.13), we find

$$\begin{aligned} & \int_s^{s+1} \int_{\{x: x \in \Omega \mid w_{mt}(t, x) > M_\delta\}} f_1(w_{mt}(t, x))(w_{mt}(t, x) - M_\delta) dx dt \\ & + \int_s^{s+1} \int_{\{x: x \in \Omega \mid w_{mt}(t, x) < -M_\delta\}} f_1(w_{mt}(t, x))(w_{mt}(t, x) + M_\delta) dx dt \leq c_{11}\delta, \quad \forall s \geq T_\delta, \end{aligned}$$

and consequently

$$\int_s^{s+1} \int_{\{x: x \in \Omega \mid |w_{mt}(t, x)| > 2M_\delta\}} f_1(w_{mt}(t, x))w_{mt}(t, x) dx dt \leq 2c_{11}\delta, \quad \forall s \geq T_\delta. \quad (4.14)$$

Now, denote $w_m^\tau(t, x) = w_m(t + \tau, x)$, $f_1^M(y) = \begin{cases} f_1(y), & |y| \leq M \\ 0, & |y| > M \end{cases}$, $\Gamma_\varepsilon(t, x) = (1 + \varepsilon + \lambda_1)w_{mt}^\tau(t, x) + w_m^\tau(t, x) - g(w_m^\tau(t, x)) - f_1^M(w_{mt}^\tau(t, x)) + h(x)$ and $\Phi(y) = f_1(y) - f_1^M(y) + \varepsilon y$, where $M \geq 1$ and $\varepsilon \in (0, 1)$. Decompose $w_m^\tau(t)$ as $w_m^\tau(t) = v_{\varepsilon m}^\tau(t) + u_{\varepsilon m}^\tau(t)$, where $v_{\varepsilon m}^\tau(t)$ and $u_{\varepsilon m}^\tau(t)$ are solutions of the problems

$$\begin{cases} v_{\varepsilon m}^{\tau} - \Delta v_{\varepsilon m}^{\tau} + v_{\varepsilon m}^{\tau} - \Delta v_{\varepsilon m}^{\tau} = \Gamma_{\varepsilon}(t, x) & \text{in } (0, 1) \times \Omega, \\ v_{\varepsilon m}^{\tau} = 0 & \text{on } (0, 1) \times \partial\Omega, \\ v_{\varepsilon m}^{\tau}(0, \cdot) = w_m^{\tau}(0), \quad v_{\varepsilon m}^{\tau}(0, \cdot) = w_{mt}^{\tau}(0) & \text{in } \Omega, \end{cases}$$

and

$$\begin{cases} u_{\varepsilon m}^{\tau} - \Delta u_{\varepsilon m}^{\tau} + u_{\varepsilon m}^{\tau} - \Delta u_{\varepsilon m}^{\tau} = -\Phi_{\varepsilon}(w_{mt}^{\tau}) & \text{in } (0, 1) \times \Omega, \\ u_{\varepsilon m}^{\tau} = 0 & \text{on } (0, 1) \times \partial\Omega, \\ u_{\varepsilon m}^{\tau}(0, \cdot) = 0, \quad u_{\varepsilon m}^{\tau}(0, \cdot) = 0 & \text{in } \Omega. \end{cases}$$

By using the techniques of the proof of Lemma 3.2, one can show that

$$\|v_{\varepsilon m}^{\tau}(1) - e^{-1}w_m^{\tau}(0)\|_{H^s(\Omega)} \leq \frac{\widehat{c}_1}{2 - s}(1 + \|f_1\|_{C[-M, M]}), \quad \forall s \in [0, 2). \quad (4.15)$$

Also, using the arguments of the proof of Lemma 3.3, we have

$$\begin{aligned} & \|u_{\varepsilon m}^{\tau}(t)\|_{L^\infty(\Omega)} \leq \frac{2\varepsilon}{(2 - \alpha)(1 - \alpha)} \\ & + \frac{1}{4\pi\alpha} \left(\int_{\Phi^{-1}(\varepsilon)}^{\infty} \frac{\Phi'(\nu)}{\nu\Phi(\nu)} d\nu + \int_{-\infty}^{\Phi^{-1}(-\varepsilon)} \frac{\Phi'(\nu)}{\nu\Phi(\nu)} d\nu \right) \int_0^1 \langle \Phi(w_{mt}^{\tau}(s)), w_{mt}^{\tau}(s) \rangle ds, \quad \forall t \in [0, 1], \end{aligned} \quad (4.16)$$

where $\alpha \in (0, 1)$. By the definition of Φ , it follows that

$$\Phi^{-1}(\varepsilon) = 1 \text{ and } \Phi^{-1}(-\varepsilon) = -1.$$

Hence,

$$\begin{aligned} & \int_{\Phi^{-1}(\varepsilon)}^{\infty} \frac{\Phi'(\nu)}{\nu\Phi(\nu)} d\nu + \int_{-\infty}^{\Phi^{-1}(-\varepsilon)} \frac{\Phi'(\nu)}{\nu\Phi(\nu)} d\nu = 2 \int_1^M \frac{1}{v^2} dv + \int_M^{\infty} \frac{f_1'(v) + \varepsilon}{v(f_1(v) + \varepsilon v)} dv \\ & + \int_{-\infty}^{-M} \frac{f_1'(v) + \varepsilon}{v(f_1(v) + \varepsilon v)} dv \leq 2 \int_1^{\infty} \frac{1}{v^2} dv + \int_1^{\infty} \frac{f_1'(v)}{v f_1(v)} dv + \int_{-\infty}^{-1} \frac{f_1'(v)}{v f_1(v)} dv \leq \widehat{c}_2. \end{aligned}$$

Taking into account the last estimate in (4.16), we obtain

$$\begin{aligned} \|u_{\varepsilon m}^{\tau}(t)\|_{L^{\infty}(\Omega)} & \leq \varepsilon \widehat{c}_3 + \int_0^1 \int_{\{x: x \in \Omega \mid |w_{mt}^{\tau}(t, x)| > M\}} f_1(w_{mt}^{\tau}(t, x)) w_{mt}^{\tau}(t, x) dx dt \\ & = \varepsilon \widehat{c}_3 + \int_{\tau}^{\tau+1} \int_{\{x: x \in \Omega \mid |w_{mt}(s, x)| > M\}} f_1(w_{mt}(s, x)) w_{mt}(s, x) dx dt, \quad \forall t \in [0, 1] \text{ and } \forall \tau \geq 0. \end{aligned} \quad (4.17)$$

Thus, by (4.14)-(4.17), we conclude that for any $\delta > 0$ there exist $\widetilde{M}_{\delta} > 0$ and $\widetilde{T}_{\delta} > 0$ such that

$$w_m(\tau + 1) - e^{-1} w_m(\tau) \in O_{\delta}^{\infty}(B(0, r(\widetilde{M}_{\delta}, s))), \quad \forall \tau \geq \widetilde{T}_{\delta},$$

where $r(M_{\delta}, s) = \frac{\widehat{c}_1}{2-s}(1 + \|f_1\|_{C[-M, M]})$, $B(0, r) = \{u : u \in H^s(\Omega), \|u\|_{H^s(\Omega)} < r\}$, $s \in (1, 2)$, and $O_{\delta}^{\infty}(B(0, r))$ is δ -neighbourhood of $B(0, r)$ in $L^{\infty}(\Omega)$. Since

$$\begin{aligned} & w_m(\tau + n) - e^{-n} w_m(\tau) \\ & = w_m(\tau + n) - e^{-1} w_m(\tau + n - 1) + e^{-1}(w_m(\tau + n - 1) - e^{-1} w_m(\tau + n - 2)) + \dots \\ & \quad + e^{-n+1}(w_m(\tau + 1) - e^{-1} w_m(\tau)), \end{aligned}$$

by the last conclusion, we have

$$w_m(\tau + n) - e^{-n} w_m(\tau) \in O_{n\delta}^{\infty}(B(0, nr(\widetilde{M}_{\delta}, s))), \quad \forall \tau \geq \widetilde{T}_{\delta}.$$

By (4.1), for any $\varepsilon > 0$ there exists $n_{\varepsilon} \in \mathbb{N}$ such that

$$e^{-n_{\varepsilon}} \|w_m(\tau)\|_{L^{\infty}(\Omega)} < \frac{\varepsilon}{3}, \quad \forall \tau \geq 0 \text{ and } \forall m \in \mathbb{N}.$$

Choosing $\delta = \frac{\varepsilon}{3n_{\varepsilon}}$, by the last two relations, we get

$$w_m(\tau + n_{\varepsilon}) \in O_{\frac{2\varepsilon}{3}}^{\infty}(B(0, r_{\varepsilon}(s))), \quad \forall \tau \geq \widetilde{T}_{\delta} \text{ and } \forall s \in (1, 2). \quad (4.18)$$

where $r_{\varepsilon}(s) = n_{\varepsilon} r(\widetilde{M}_{\delta}, s)$. By the compact embedding $H^s(\Omega) \subset L^{\infty}(\Omega)$, for $s > 1$, it follows that $B(0, r_{\varepsilon}(s))$ is relatively compact subset of $L^{\infty}(\Omega)$. Hence, by (4.18), the set $\{w_m(\tau + n_{\varepsilon})\}_{\tau \geq \widetilde{T}_{\delta}}$ and particularly, $\{w_m(t_m)\}_{m=1}^{\infty}$ has a finite ε -net in $L^{\infty}(\Omega)$. Since ε is arbitrary positive number, we obtain relatively compactness of $\{w_m(t_m)\}_{m=1}^{\infty}$ in $L^{\infty}(\Omega)$, which, together with the compactness proved in *Step I*, completes the proof. \square

Since, by (2.2) and (3.14), the problem (2.1) admits a strict Lyapunov function $L(w(t)) = \frac{1}{2} \|w_t(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla w(t)\|_{L^2(\Omega)}^2 + \langle G(w(t)), 1 \rangle - \langle h, w(t) \rangle$, by Lemma 4.1-4.2 and [24, Corollary 2.29], we have Theorem 2.2.

APPENDIX A.

Lemma A.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Then for $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$, there exists a sequence $\{u_n\} \subset C_0^\infty(\Omega)$ such that*

$$\lim_{n \rightarrow \infty} \|u_0 - u_n\|_{H_0^1(\Omega)} = 0 \quad \text{and} \quad \sup_n \|u_n\|_{C(\overline{\Omega})} \leq \|u_0\|_{L^\infty(\Omega)}.$$

Proof. By the definition of $H_0^1(\Omega)$, there exists $\{v_n\} \subset C_0^\infty(\Omega)$ such that

$$v_n \rightarrow u_0 \text{ strongly in } H_0^1(\Omega).$$

Then there exists a subsequence $\{v_{n_k}\} \subset C_0^\infty(\Omega)$ such that

$$v_{n_k} \rightarrow u_0 \text{ a.e. in } \Omega. \quad (\text{A.1})$$

Let $M_0 = \|u_0\|_{L^\infty(\Omega)}$. Setting $w_k(x) := \begin{cases} -M_0, & v_{n_k}(x) < -M_0, \\ v_{n_k}, & |v_{n_k}(x)| \leq M_0, \\ M_0, & v_{n_k}(x) > M_0 \end{cases}$ we have $w_k \in H_0^1(\Omega) \cap C(\overline{\Omega})$, $\sup_k \|w_k\|_{C(\overline{\Omega})} \leq M_0$, $\overline{\{x : x \in \Omega, w_k \neq 0\}} \subset \Omega$ and

$$\begin{aligned} \limsup_{k \rightarrow \infty} \|\nabla(u_0 - w_k)\|_{L^2(\Omega)}^2 &= \limsup_{k \rightarrow \infty} \int_{\{x : |v_{n_k}(x)| > M_0\}} |\nabla u_0(x)|^2 dx \\ &\leq \limsup_{m \rightarrow \infty} \int_{\bigcup_{k \geq m} \{x : |v_{n_k}(x)| > M_0\}} |\nabla u_0(x)|^2 dx = \int_{\bigcap_{m \geq m} \bigcup_{k \geq m} \{x : |v_{n_k}(x)| > M_0\}} |\nabla u_0(x)|^2 dx. \end{aligned} \quad (\text{A.2})$$

Let A be a subset of Ω , where a sequence $\{v_{n_k}\}$ does not converge to u_0 . Then we have

$$\bigcap_m \bigcup_{k \geq m} \{x : |v_{n_k}(x)| > M_0\} \subset \{x : |u_0(x)| \geq M_0\} \cup A. \quad (\text{A.3})$$

Since, by (A.1), $\text{mes}(A) = 0$, taking into account (A.3) in (A.2), we find that

$$\begin{aligned} \limsup_{k \rightarrow \infty} \|\nabla(u_0 - w_k)\|_{L^2(\Omega)}^2 &\leq \int_{\{x : |u_0(x)| \geq M_0\} \cup A} |\nabla u_0(x)|^2 dx \\ &= \int_{\{x : |u_0(x)| \geq M_0\}} |\nabla u_0(x)|^2 dx. \end{aligned}$$

Also, considering

$$\text{mes}(\{x : |u_0(x)| > M_0\}) = 0 \quad \text{and} \quad \int_{\{x : |u_0(x)| = M_0\}} |\nabla u_0(x)|^2 dx = 0,$$

in the last inequality, we obtain

$$w_k \rightarrow u_0 \text{ strongly in } H_0^1(\Omega).$$

Now, to complete the proof, it is sufficient to show that for any $w \in H_0^1(\Omega) \cap C(\overline{\Omega})$ with $\overline{\{x : x \in \Omega, w \neq 0\}} \subset \Omega$, there exists $\{u_n\} \subset C_0^\infty(\Omega)$ such that

$$\lim_{n \rightarrow \infty} \|w - u_n\|_{H_0^1(\Omega)} = 0 \quad \text{and} \quad \sup_n \|u_n\|_{C(\overline{\Omega})} \leq \|w\|_{C(\overline{\Omega})} \quad (\text{A.4})$$

Denoting $\overline{w}(x) = \begin{cases} w(x), & x \in \Omega \\ 0, & x \in \mathbb{R}^N \setminus \Omega \end{cases}$ and $u_n(x) = (\rho_n * \overline{w})(x)$, we have $\overline{\{x : x \in \Omega, u_n \neq 0\}} \subset \Omega$ for sufficiently large n , $\{u_n\} \subset C^\infty(\mathbb{R}^N)$,

$$\lim_{n \rightarrow \infty} \|\overline{w} - u_n\|_{H^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \sup_n \|u_n\|_{L^\infty(\mathbb{R}^2)} \leq \|w\|_{C(\overline{\Omega})},$$

where $*$ denotes the convolution, $\rho_n(x) = \begin{cases} Kn^N e^{-\frac{1}{1-n^2|x|^2}}, & |x| < \frac{1}{n}, \\ 0, & |x| \geq \frac{1}{n} \end{cases}$, $n \in \mathbb{N}$ and $K^{-1} = \int_{\{x: |x| < 1\}} e^{-\frac{1}{1-|x|^2}} dx$. So, the restriction of the sequence $\{u_n\}$ to Ω satisfies (A.4), for sufficiently large n . \square

Lemma A.2. *Let $Q \subset \mathbb{R}^N$ be a measurable set with finite measure and φ a continuous function on \mathbb{R} such that $\varphi(s)s \geq 0$ for every $s \in \mathbb{R}$. If*

$$u_n \rightarrow u \text{ a.e. in } Q \text{ and } \sup_n \int_Q \varphi(u_n(x))u_n(x)dx < \infty,$$

then

$$\lim_{n \rightarrow \infty} \|\varphi(u_n) - \varphi(u)\|_{L^1(Q)} = 0. \quad (\text{A.5})$$

Proof. By the continuity of φ , we have

$$\begin{aligned} \int_Q |\varphi(u_n(x))| dx &\leq \int_{\{x: x \in Q, |u_n(x)| \leq 1\}} |\varphi(u_n(x))| dx \\ &+ \int_{\{x: x \in Q, |u_n(x)| > 1\}} |\varphi(u_n(x))| dx \leq \|\varphi\|_{C[-1,1]} \text{mes}(Q) \\ &+ \int_Q \varphi(u_n(x))u_n(x)dx. \end{aligned}$$

So, $\varphi(u_n) \in L^1(Q)$. Applying Fatou's lemma, we have

$$\int_Q \varphi(u(x))u(x)dx \leq \liminf_{n \rightarrow \infty} \int_Q \varphi(u_n(x))u_n(x)dx < \infty,$$

which, as shown above, yields $\varphi(u) \in L^1(Q)$. Also, by Egorov's theorem, for any $\varepsilon > 0$, there exists $Q_\varepsilon \subset Q$ such that $\text{mes}(Q \setminus Q_\varepsilon) < \varepsilon$ and

$$u_n \rightarrow u \text{ uniformly in } Q_\varepsilon.$$

Now, denote $A_k = \{x : x \in Q, |u(x)| \leq k\}$ and $A_{nk} = \{x : x \in Q, |u_n(x)| \leq k\}$, for $k > 1$. By the last approximation, we get

$$\varphi(u_n) \rightarrow \varphi(u) \text{ uniformly in } Q_\varepsilon \cap A_k. \quad (\text{A.6})$$

Since, for sufficiently large n ,

$$\begin{aligned} \int_{Q_\varepsilon \setminus A_k} |\varphi(u_n(x)) - \varphi(u(x))| dx &\leq \frac{1}{k-1} \int_Q \varphi(u_n(x))u_n(x)dx \\ &+ \frac{1}{k} \int_Q \varphi(u(x))u(x)dx, \end{aligned}$$

using (A.6), we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_Q |\varphi(u_n(x)) - \varphi(u(x))| dx &\leq \limsup_{n \rightarrow \infty} \int_{Q_\varepsilon \cap A_k} |\varphi(u_n(x)) - \varphi(u(x))| dx \\ &+ \limsup_{n \rightarrow \infty} \int_{Q_\varepsilon \setminus A_k} |\varphi(u_n(x)) - \varphi(u(x))| dx + \limsup_{n \rightarrow \infty} \int_{Q \setminus Q_\varepsilon} |\varphi(u_n(x)) - \varphi(u(x))| dx \\ &\leq \frac{1}{k-1} \limsup_{n \rightarrow \infty} \int_Q \varphi(u_n(x))u_n(x)dx + \frac{1}{k} \int_Q \varphi(u(x))u(x)dx \end{aligned}$$

$$+\limsup_{n \rightarrow \infty} \int_{Q \setminus Q_\varepsilon} |\varphi(u_n(x)) - \varphi(u(x))| dx.$$

Passing to the limit as $k \rightarrow \infty$ and $\varepsilon \rightarrow 0$, we find

$$\limsup_{n \rightarrow \infty} \int_Q |\varphi(u_n(x)) - \varphi(u(x))| dx \leq \limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \int_{Q \setminus Q_\varepsilon} |\varphi(u_n(x)) - \varphi(u(x))| dx. \quad (\text{A.7})$$

Now, let us estimate the right hand side of (A.7).

$$\begin{aligned} \int_{Q \setminus Q_\varepsilon} |\varphi(u_n(x)) - \varphi(u(x))| dx &\leq \int_{Q \setminus Q_\varepsilon} |\varphi(u_n(x))| dx + \int_{Q \setminus Q_\varepsilon} |\varphi(u(x))| dx \\ &\leq \frac{1}{k} \int_Q \varphi(u_n(x)) u_n(x) dx + \frac{1}{k} \int_Q \varphi(u(x)) u(x) dx \\ &\quad + \int_{(Q \setminus Q_\varepsilon) \cap A_{nk}} |\varphi(u_n(x))| dx + \int_{(Q \setminus Q_\varepsilon) \cap A_k} |\varphi(u(x))| dx \\ &\leq \frac{1}{k} \int_Q \varphi(u_n(x)) u_n(x) dx + \frac{1}{k} \int_Q \varphi(u(x)) u(x) dx \\ &\quad + 2 \|\varphi\|_{C[-k, k]} \text{mes}(Q \setminus Q_\varepsilon) \end{aligned}$$

and consequently

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \int_{Q \setminus Q_\varepsilon} |\varphi(u_n(x)) - \varphi(u(x))| dx &\leq \frac{1}{k} \int_Q \varphi(u(x)) u(x) dx \\ &\quad + \frac{1}{k} \limsup_{n \rightarrow \infty} \int_Q \varphi(u_n(x)) u_n(x) dx. \end{aligned}$$

Passing to the limit as $k \rightarrow \infty$, we obtain

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \int_{Q \setminus Q_\varepsilon} |\varphi(u_n(x)) - \varphi(u(x))| dx = 0,$$

which, together with (A.7), gives us (A.5). \square

Lemma A.3. *Let $\Omega \subset R^N$ be a bounded domain with smooth boundary and $h \in L^1((0, T) \times \Omega)$. If $u \in C_s(0, T; L^1(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ with $\lim_{t \searrow 0} \|u(t)\|_{L^1(\Omega)} = 0$, is a solution of*

$$\begin{cases} u_t(t, x) - \Delta u(s, x) = h(t, x), & (t, x) \in (0, T) \times \Omega, \\ u(t, x) = 0, & (t, x) \in ((0, T) \times \partial\Omega) \cup (\{0\} \times \Omega) \end{cases} \quad (\text{A.8})$$

then

$$|u(t, x)| \leq \frac{1}{(4\pi)^{\frac{N}{2}}} \int_0^t \frac{1}{(t-s)^{\frac{N}{2}}} \int_\Omega e^{-\frac{|x-y|^2}{4(t-s)}} |h(s, y)| dy ds, \quad \text{a.e. in } (0, T) \times \Omega, \quad (\text{A.9})$$

where $C_s(0, T; L^1(\Omega)) = \left\{ u : u \in L^\infty(0, T; L^1(\Omega)), \int_\Omega \varphi(x) u(\cdot, x) dx \in C[0, T] \text{ for every } \varphi \in L^\infty(\Omega) \right\}$.

Proof. Let $\theta_{tn}(\tau) = \begin{cases} 0, & \forall \tau \in [0, \frac{1}{2n}), \\ 2n(\tau - \frac{1}{2n}), & \forall \tau \in [\frac{1}{2n}, \frac{1}{n}), \\ 1, & \forall \tau \in [\frac{1}{n}, t - \frac{1}{n}), \\ 2n(t - \frac{1}{2n} - \tau), & \forall \tau \in [t - \frac{1}{n}, t - \frac{1}{2n}], \\ 0, & \forall \tau \in (t - \frac{1}{2n}, t] \end{cases}$. Also denote $\varphi_m(x) = \begin{cases} -1, & x < -\frac{1}{m}, \\ mx, & |x| \leq \frac{1}{m}, \\ 1, & x > \frac{1}{m} \end{cases}$, $\delta_k(t) = \begin{cases} Mk^2 e^{-\frac{1}{1-k^2|t|^2}}, & |t| < \frac{1}{k}, \\ 0, & |t| \geq \frac{1}{k} \end{cases}$, where $m, k \in \mathbb{N}$ and $M^{-1} = \int_{-1}^1 e^{-\frac{1}{1-t^2}} dt$. Testing (A.8)₁ by $(\theta_{\tau n}(t)\delta_k(s-t))$ on $(0, \tau)$, we get

$$\begin{aligned} & \frac{\partial}{\partial s}(\theta_{\tau n}u(\cdot, x) * \delta_k)(s) - (u(\cdot, x)\theta'_{\tau n} * \delta_k)(s) - \Delta(\theta_{\tau n}u(\cdot, x) * \delta_k)(s) \\ &= (h(\cdot, x)\theta_{\tau n} * \delta_k)(s), \quad (s, x) \in (0, \tau) \times \Omega, \quad \tau \in (\frac{1}{2n}, T). \end{aligned}$$

Testing the last equation by $\varphi_m((\theta_{\tau n}u) * \delta_k)(s, x)$ on $(\frac{1}{3n}, \tau) \times \Omega$, we obtain

$$\begin{aligned} & \int_{\Omega} \Phi_m((\theta_{\tau n}u(\cdot, x)) * \delta_k)(\tau) dx - \int_{\Omega} \Phi_m((\theta_{\tau n}u(\cdot, x)) * \delta_k)(\frac{1}{3n}) dx \\ & - \int_{\frac{1}{3n}}^t \int_{\Omega} (u(\cdot, x)\theta'_{\tau n} * \delta_k)(s) \varphi_m((\theta_{\tau n}u(\cdot, x)) * \delta_k)(s) dx ds \\ & \leq \int_{\frac{1}{3n}}^t \int_{\Omega} |(h(\cdot, x)\theta_{\tau n} * \delta_k)(s) \varphi_m((\theta_{\tau n}u(\cdot, x)) * \delta_k)(s)| dx ds, \end{aligned} \quad (\text{A.10})$$

where $\Phi_m(y) = \int_0^y \varphi_m(x) dx$. By the definition of θ_{tn} , δ_k and φ_m , we have

$$\left\{ \begin{aligned} & \lim_{k \rightarrow \infty} \int_{\Omega} \Phi_m((\theta_{\tau n}u(\cdot, x)) * \delta_k)(\frac{1}{3n}) dx = 0, \\ & \lim_{k \rightarrow \infty} \int_{\Omega} \Phi_m((\theta_{\tau n}u(\cdot, x)) * \delta_k)(\tau) dx = 0, \\ & \lim_{k \rightarrow \infty} \int_{\frac{1}{3n}}^{\tau} \int_{\Omega} (u(\cdot, x)\theta'_{\tau n} * \delta_k)(s) \varphi_m((\theta_{\tau n}u(\cdot, x)) * \delta_k)(s) dx ds \\ & \quad = \int_{\frac{1}{2n}}^{\tau} \int_{\Omega} \theta'_{\tau n}(s) u(s, x) \varphi_m(\theta_{\tau n}(s)u(s, x)) dx ds, \\ & \limsup_{k \rightarrow \infty} \int_{\frac{1}{3n}}^{\tau} \int_{\Omega} |(h(\cdot, x)\theta_{\tau n} * \delta_k)(s) \varphi_m((\theta_{\tau n}u(\cdot, x)) * \delta_k)(s)| dx ds \\ & \leq \limsup_{k \rightarrow \infty} \int_{\frac{1}{3n}}^{\tau} \int_{\Omega} |(h(\cdot, x)\theta_{\tau n} * \delta_k)(s)| dx ds = \int_{\frac{1}{2n}}^{\tau} \int_{\Omega} |h(s, x)\theta_{\tau n}(s)| dx ds. \end{aligned} \right.$$

Thus, passing to the limit in (A.10) as $k \rightarrow \infty$, we obtain

$$- \int_{\frac{1}{2n}}^{\tau} \int_{\Omega} u(s, x) \theta'_{\tau n}(s) \varphi_m(\theta_{\tau n}(s)u(s, x)) dx ds \leq \int_{\frac{1}{2n}}^{\tau} \int_{\Omega} |h(s, x)\theta_{\tau n}(s)| dx ds$$

Now, take the limit in the last inequality as $m \rightarrow \infty$, we find

$$2n \int_{\tau - \frac{1}{n}}^{\tau - \frac{1}{2n}} \int_{\Omega} \|u(s)\|_{L^1(\Omega)} ds - 2n \int_{\frac{1}{2n}}^{\frac{1}{n}} \|u(s)\|_{L^1(\Omega)} ds \leq \int_0^{\tau} \|h(s)\|_{L^1(\Omega)} ds. \quad (\text{A.11})$$

Since $u \in C_s(0, T; L_1(\Omega))$, by the weak lower semi-continuity of the norm, it follows that

$$\liminf_{n \rightarrow \infty} 2n \int_{t-\frac{1}{n}}^{t-\frac{1}{2n}} \int_{\Omega} \|u(\tau)\|_{L^1(\Omega)} d\tau \geq \|u(t)\|_{L^1(\Omega)}.$$

Also, by the condition $\lim_{t \searrow 0} \|u(t)\|_{L^1(\Omega)} = 0$, we have

$$\lim_{n \rightarrow \infty} 2n \int_{\frac{1}{2n}}^{\frac{1}{n}} \|u(\tau)\|_{L^1(\Omega)} d\tau = 0.$$

Hence, passing to the limit in (A.11) as $n \rightarrow \infty$, we get

$$\|u(t)\|_{L^1(\Omega)} \leq \|h\|_{L^1((0,t) \times \Omega)}, \quad \forall t \in [0, T]. \quad (\text{A.12})$$

Define $H(t, x) = \begin{cases} h(t, x), & (t, x) \in (0, T) \times \Omega \\ 0, & (t, x) \in R^{N+1} \setminus (0, T) \times \Omega \end{cases}$, $H_m(t, x) = (H * \rho_m)(t, x)$ and $\overline{H}_m(t, x) = (|H| * \rho_m)(t, x)$. Since $H_m \in C_0(R^{N+1})$ and $\overline{H}_m \in C_0(R^{N+1})$, the problems

$$\begin{cases} u_{mt}(t, x) - \Delta u_m(s, x) = H_m(t, x), & (t, x) \in (0, T) \times \Omega, \\ u_m(t, x) = 0, & (t, x) \in ((0, T) \times \partial\Omega) \cup (\{0\} \times \Omega) \end{cases}$$

and

$$\begin{cases} v_{mt}(t, x) - \Delta v_m(s, x) = \overline{H}_m(t, x), & (t, x) \in (0, T) \times R^N, \\ v_m(t, x) = 0, & (t, x) \in \{0\} \times R^N \end{cases}$$

have unique smooth classical solutions. By (A.12), it follows that

$$\|u(t) - u_m(t)\|_{L^1(\Omega)} \leq \|H - H_m\|_{L^1((0,t) \times \Omega)}, \quad \forall t \in [0, T]. \quad (\text{A.13})$$

On the other hand, applying Duhamel's principle (see [25, p. 49]), we get

$$v_m(t, x) = \frac{1}{(4\pi)^{\frac{N}{2}}} \int_0^t \frac{1}{(t-s)^{\frac{N}{2}}} \int_{R^N} e^{-\frac{|x-y|^2}{4(t-s)}} \overline{H}_m(s, y) dy ds. \quad (\text{A.14})$$

Denoting $w_m(t, x) = v_m(t, x) - u_m(t, x)$, we have

$$\begin{cases} w_{mt}(t, x) - \Delta w_m(s, x) \geq 0, & (t, x) \in (0, T) \times \Omega, \\ w_m(t, x) \geq 0, & (t, x) \in (0, T) \times \partial\Omega, \\ w_m(t, x) = 0, & (t, x) \in \{0\} \times \Omega. \end{cases}$$

So, by the maximum principle, it follows that

$$w_m(t, x) \geq 0 \text{ or } v_m(t, x) \geq u_m(t, x), \quad \forall (t, x) \in [0, T] \times \overline{\Omega}.$$

By the similar way, one can show that

$$u_m(t, x) \geq -v_m(t, x), \quad \forall (t, x) \in [0, T] \times \overline{\Omega}.$$

Hence,

$$|u_m(t, x)| \leq v_m(t, x), \quad \forall (t, x) \in [0, T] \times \overline{\Omega},$$

which, together with (A.14), yields

$$|u_m(t, x)| \leq \frac{1}{(4\pi)^{\frac{N}{2}}} \int_0^t \frac{1}{(t-s)^{\frac{N}{2}}} \int_{R^N} e^{-\frac{|x-y|^2}{4(t-s)}} \overline{H}_m(s, y) dy ds, \quad \forall (t, x) \in [0, T] \times \overline{\Omega}. \quad (\text{A.15})$$

Since

$$\left\| \int_0^t \frac{1}{(t-s)^{\frac{N}{2}}} \int_{R^N} e^{-\frac{|x-y|^2}{4(t-s)}} \overline{H}_m(s, y) dy ds - \int_0^t \frac{1}{(t-s)^{\frac{N}{2}}} \int_{\Omega} e^{-\frac{|x-y|^2}{4(t-s)}} |h(s, y)| dy ds \right\|_{L^1(\Omega)}$$

$$\begin{aligned}
&\leq \int_{R^N} \int_0^t \frac{1}{(t-s)^{\frac{N}{2}}} \int_{R^N} e^{-\frac{|x-y|^2}{4(t-s)}} |\overline{H}_m(s, y) - |H(s, y)|| dy ds dx \\
&= \int_0^t \int_{R^N} |\overline{H}_m(s, y) - |H(s, y)|| \frac{1}{(t-s)^{\frac{N}{2}}} \int_{R^N} e^{-\frac{|x-y|^2}{4(t-s)}} dx dy ds \\
&= (4\pi)^{\frac{N}{2}} \int_0^t \int_{R^N} |\overline{H}_m(s, y) - |H(s, y)|| dy ds \\
&= (4\pi)^{\frac{N}{2}} \| |H| - |H| * \rho_m \|_{L^1((0,t) \times \Omega)}, \quad \forall t \in [0, T],
\end{aligned} \tag{A.16}$$

passing to the limit in (A.15) and taking into account (A.13) and (A.16), we obtain (A.9). \square

Lemma A.4. *Let $Q \subset R^N$ be a measurable set of finite measure and φ an increasing continuous function such that $\varphi(0) = 0$. Further assume that $v_i \in L^\infty(0, T; L^\infty(Q))$ and $v_{it} \in L^1(0, T; L^1(Q))$, $i = 1, 2$. If*

$$\int_0^T \int_Q (|\varphi(v_{it}(t, x))| + |\varphi(-v_{it}(t, x))|) |v_{it}(t, x)| dx dt < \infty, \quad i = 1, 2,$$

then

$$\liminf_{h \searrow 0} \int_s^t \int_s^\sigma \int_Q (\varphi(v_{2t}(\tau, x)) - \varphi(v_{1t}(\tau, x))) \frac{w(\tau + h, x) - w(\tau - h, x)}{2h} dx d\tau d\sigma \geq 0, \tag{A.17}$$

for every $[s, t] \subset (0, T)$, where $w(t, x) = v_2(t, x) - v_1(t, x)$.

Proof. By the conditions of the lemma, it follows that

$$\begin{aligned}
&\int_0^T \int_Q |\varphi(v_{it}(t, x))| dx dt = \int_0^T \int_{\{x: x \in Q, |v_{it}(t, x)| > 1\}} |\varphi(v_{it}(t, x))| dx dt \\
&\quad + \int_0^T \int_{\{x: x \in Q, |v_{it}(t, x)| \leq 1\}} |\varphi(v_{it}(t, x))| dx dt \\
&\leq \int_0^T \int_{\{x: x \in Q, |v_{it}(t, x)| > 1\}} |\varphi(v_{it}(t, x))| |v_{it}(t, x)| dx dt \\
&\quad + \text{mes}(Q) \|\varphi\|_{C[-1, 1]} < \infty, \quad i = 1, 2.
\end{aligned}$$

Therefore, the integral in (A.17) is well defined.

Now, let us denote $M_\varphi(t) = \int_0^t \varphi(s) ds$, $N_\varphi(t) = \int_0^t \varphi^{-1}(s) ds$ and $g(t) = -\varphi(-t)$. By Young's inequality (see [26, p. 12]), we have

$$uv \leq M_\varphi(u) + N_\varphi(v), \quad \forall u, v \geq 0. \tag{A.18}$$

If $u < 0$ and $v < 0$, then again by Young's inequality,

$$uv = -u(-v) \leq M_g(-u) + N_g(-v) = M_\varphi(u) + N_\varphi(v).$$

Since the right hand side of (A.18) is nonnegative for all $u, v \in R$, in the case $uv < 0$, this inequality is trivial. Hence, the inequality (A.18) holds for all $u, v \in R$. Therefore we conclude that

$$uv = -u(-v) \geq -M_\varphi(u) - N_\varphi(-v), \quad \forall u, v \in R. \tag{A.19}$$

Denote $\varphi_M(x) = \begin{cases} -M, & x < -M, \\ \varphi(x), & |x| \leq M, \\ M, & x > M \end{cases}$. Since, by definition,

$$\lim_{h \searrow 0} \left\| \frac{w(\cdot + h) - w(\cdot - h)}{2h} - w_t(\cdot) \right\|_{L^1((0,t) \times Q)} = 0, \quad \forall t \in (0, T),$$

we have

$$\frac{w(t+h, x) - w(t-h, x)}{2h} \rightarrow w_t(t, x) \text{ in the measure as } h \searrow 0.$$

Hence, applying Lebesgue's convergence theorem and taking into account the monotonicity of φ_M , we get

$$\begin{aligned} & \lim_{h \searrow 0} \int_s^t \int_s^\sigma \int_Q (\varphi_M(v_{2t}(\tau, x)) - \varphi_M(v_{1t}(\tau, x))) \frac{w(\tau+h, x) - w(\tau-h, x)}{2h} dx d\tau d\sigma \\ &= \int_s^t \int_s^\sigma \int_Q (\varphi_M(v_{2t}(\tau, x)) - \varphi_M(v_{1t}(\tau, x))) w_t(\tau, x) dx d\tau d\sigma \geq 0 \end{aligned}$$

and consequently

$$\begin{aligned} & \liminf_{h \searrow 0} \int_s^t \int_s^\sigma \int_Q (\varphi(v_{2t}(\tau, x)) - \varphi(v_{1t}(\tau, x))) \frac{w(\tau+h, x) - w(\tau-h, x)}{2h} dx d\tau d\sigma \\ & \geq \liminf_{h \searrow 0} \int_s^t \int_s^\sigma \int_Q (\varphi(v_{2t}(\tau, x)) - \varphi_M(v_{2t}(\tau, x))) \frac{v_2(\tau+h, x) - v_2(\tau-h, x)}{2h} dx d\tau \\ & + \liminf_{h \searrow 0} \int_s^t \int_s^\sigma \int_Q (\varphi(v_{2t}(\tau, x)) - \varphi_M(v_{2t}(\tau, x))) \frac{v_1(\tau-h, x) - v_1(\tau+h, x)}{2h} dx d\tau d\sigma \\ & + \liminf_{h \searrow 0} \int_s^t \int_s^\sigma \int_Q (\varphi(v_{1t}(\tau, x)) - \varphi_M(v_{1t}(\tau, x))) \frac{v_1(\tau+h, x) - v_1(\tau-h, x)}{2h} dx d\tau d\sigma \\ & + \liminf_{h \searrow 0} \int_s^t \int_s^\sigma \int_Q (\varphi(v_{1t}(\tau, x)) - \varphi_M(v_{1t}(\tau, x))) \frac{v_2(\tau-h, x) - v_2(\tau+h, x)}{2h} dx d\tau d\sigma \\ & =: I_1^M(s, t) + I_2^M(s, t) + I_3^M(s, t) + I_4^M(s, t), \quad \forall [s, t] \subset (0, T) \text{ and } \forall M > 0. \end{aligned} \quad (\text{A.20})$$

Now, let us estimate each $I_i^M(s, t)$ ($i = 1, 2, 3, 4$). By (A.19) and Jensen's inequality for convex functions (see [26, p. 62]), we have

$$\begin{aligned} I_1^M(t) &= \liminf_{h \searrow 0} \int_s^t \int_s^\sigma \int_Q (\varphi(v_{2t}(\tau, x)) - \varphi_m(v_{2t}(\tau, x))) \frac{v_2(\tau+h, x) - v_2(\tau-h, x)}{2h} dx d\tau d\sigma \\ &\geq - \int_s^t \int_s^\sigma \int_{\{x: x \in Q, |v_{2t}(\tau, x)| > M\}} N_\varphi(\varphi(v_{2t}(\tau, x)) - \varphi_m(v_{2t}(\tau, x))) dx d\tau d\sigma \\ &\quad - \limsup_{h \searrow 0} \int_s^t \int_s^\sigma \int_{\{x: x \in Q, |v_{2t}(\tau, x)| > M\}} M_\varphi \left(\frac{1}{2} \int_{-1}^1 -v_{2t}(\tau + \mu h, x) d\mu \right) dx d\tau d\sigma \end{aligned}$$

$$\begin{aligned}
&\geq - \int_s^t \int_s^\sigma \int_{\{x: x \in Q, |v_{2t}(\tau, x)| > M\}} N_\varphi(\varphi(v_{2t}(\tau, x))) dx d\tau d\sigma \\
&- \frac{1}{2} \limsup_{h \searrow 0} \int_s^t \int_s^\sigma \int_{\{x: x \in Q, |v_{2t}(\tau, x)| > M\}} \int_{-1}^1 M_\varphi(-v_{2t}(\tau + \mu h, x)) d\mu dx d\tau d\sigma. \tag{A.21}
\end{aligned}$$

By the definition of N_φ and M_φ , we obtain

$$\begin{aligned}
&\int_s^t \int_s^\sigma \int_{\{x: x \in Q, |v_{2t}(\tau, x)| > M\}} N_\varphi(\varphi(v_{2t}(\tau, x))) dx d\tau d\sigma \\
&\leq T \int_0^T \int_{\{x: x \in Q, |v_{2t}(\tau, x)| > M\}} \varphi(v_{2t}(\tau, x)) v_{2t}(\tau, x) dx d\tau, \tag{A.22}
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{h \searrow 0} \int_s^t \int_s^\sigma \int_{\{x: x \in Q, |v_{2t}(\tau, x)| > M\}} \int_{-1}^1 M_\varphi(-v_{2t}(\tau + \mu h, x)) d\mu dx d\tau d\sigma \\
&\leq 2 \int_s^t \int_s^\sigma \int_{\{x: x \in Q, |v_{2t}(\tau, x)| > M\}} M_\varphi(-v_{2t}(\tau, x)) dx d\tau d\sigma + \\
&+ \limsup_{h \searrow 0} \int_{-1}^1 \int_s^t \int_s^\sigma \int_Q |M_\varphi(-v_{2t}(\tau + \mu h, x)) - M_\varphi(-v_{2t}(\tau, x))| dx d\tau d\sigma d\mu \\
&\leq 2 \int_s^t \int_s^\sigma \int_{\{x: x \in Q, |v_{2t}(\tau, x)| > M\}} |\varphi(-v_{2t}(\tau, x))| |v_{2t}(\tau, x)| dx d\tau d\sigma \\
&+ \limsup_{h \searrow 0} \int_{-1}^1 \int_s^t \int_s^\sigma \int_Q |M_\varphi(-v_{2t}(\tau + \mu h, x)) - M_\varphi(-v_{2t}(\tau, x))| dx d\tau d\sigma d\mu. \tag{A.23}
\end{aligned}$$

Now, to pass to the limit under last the integral, we apply Lebesgue's convergence theorem. Since $v_{2t} \in L^1(0, T; L^1(Q))$, we have (see [21, Remark 3.2])

$$\lim_{h \searrow 0} \int_s^\sigma \int_Q |v_{2t}(\tau + \mu h, x) - v_{2t}(\tau, x)| dx d\tau = 0, \quad \forall [s, \sigma] \subset (0, T),$$

which yields

$$v_{2t}(\cdot + h\mu, \cdot) \rightarrow v_{2t}(\cdot, \cdot) \text{ in measure as } h \rightarrow 0.$$

Also, for $\mu \in [-1, 1]$ and sufficiently small $h > 0$, it is easy to see that

$$\begin{aligned}
&\int_s^\sigma \int_Q |M_\varphi(-v_{2t}(\tau + \mu h, x)) - M_\varphi(-v_{2t}(\tau, x))| dx d\tau \\
&\leq \int_{s+\mu h}^{\sigma+\mu h} \int_Q |\varphi(-v_{2t}(\tau, x))| |v_{2t}(\tau, x)| dx d\tau + \int_s^\sigma \int_Q |\varphi(-v_{2t}(\tau, x))| |v_{2t}(\tau, x)| dx d\tau \\
&\leq 2 \int_0^T \int_Q |\varphi(-v_{2t}(\tau, x))| |v_{2t}(\tau, x)| dx d\tau.
\end{aligned}$$

Hence, by the Lebesgue's convergence theorem,

$$\limsup_{h \searrow 0} \int_{-1}^1 \int_s^t \int_s^\sigma \int_Q |M_\varphi(-v_{2t}(\tau + \mu h, x)) - M_\varphi(-v_{2t}(\tau, x))| dx d\tau d\sigma d\mu = 0. \quad (\text{A.24})$$

Taking into account (A.22)-(A.24) in (A.21), we get

$$\begin{aligned} I_1^M(s, t) &\geq -T \int_0^T \int_{\{x: x \in Q, |v_{2t}(\tau, x)| > M\}} \varphi(v_{2t}(\tau, x)) v_{2t}(\tau, x) dx d\tau \\ &\quad - T \int_0^T \int_{\{x: x \in Q, |v_{2t}(\tau, x)| > M\}} |\varphi(-v_{2t}(\tau, x))| |v_{2t}(\tau, x)| dx d\tau, \quad \forall [s, t] \subset (0, T), \end{aligned}$$

and consequently

$$\liminf_{M \rightarrow \infty} I_1^M(s, t) \geq 0, \quad \forall [s, t] \subset (0, T).$$

By the same way, one can show that

$$\liminf_{M \rightarrow \infty} I_i^M(s, t) \geq 0, \quad \forall [s, t] \subset (0, T), \quad i = 2, 3, 4,$$

which, together with (A.20), gives us (A.17). \square

Lemma A.5. *Let $Q \subset \mathbb{R}^N$ be a measurable set of finite measure and φ an increasing continuous function such that $\varphi(0) = 0$. Also assume that $w \in L^\infty(0, T; L^\infty(Q))$ and $w_t \in L^1(0, T; L^1(Q))$. If*

$$\int_0^T \int_Q (|\varphi(w_t(t, x))| + |\varphi(-w_t(t, x))|) |w_t(t, x)| dx dt < \infty,$$

then

$$\liminf_{h \rightarrow 0} \max_{s \leq \sigma \leq t} \int_\sigma^{\sigma+|h|} \int_Q \left| \varphi(w_t(\tau, x)) \frac{w(\tau+h, x) - w(\tau, x)}{h} \right| dx d\tau = 0,$$

for every $[s, t] \subset (0, T)$.

Proof. By using techniques of the previous lemma, we find

$$\begin{aligned} &\int_\sigma^{\sigma+|h|} \int_Q \left| \varphi(w_t(\tau, x)) \frac{w(\tau+h, x) - w(\tau, x)}{h} \right| dx d\tau \\ &= \int_\sigma^{\sigma+|h|} \int_{\{x: x \in Q, w_t(\tau, x) \geq 0\}} \varphi(w_t(\tau, x)) \left| \frac{w(\tau+h, x) - w(\tau, x)}{h} \right| dx d\tau \\ &\quad - \int_\sigma^{\sigma+|h|} \int_{\{x: x \in Q, w_t(\tau, x) < 0\}} \varphi(w_t(\tau, x)) \left| \frac{w(\tau+h, x) - w(\tau, x)}{h} \right| dx d\tau \\ &\leq \int_\sigma^{\sigma+|h|} \int_{\{x: x \in Q, w_t(\tau, x) \geq 0\}} N_\varphi(\varphi(w_t(\tau, x))) dx d\tau \\ &\quad + \int_\sigma^{\sigma+|h|} \int_{\{x: x \in Q, w_t(\tau, x) \geq 0\}} M_\varphi \left(\left| \frac{w(\tau+h, x) - w(\tau, x)}{h} \right| \right) dx d\tau \end{aligned}$$

$$\begin{aligned}
& + \int_{\sigma}^{\sigma+|h|} \int_{\{x: x \in Q, w_t(\tau, x) < 0\}} N_{\varphi}(\varphi(w_t(\tau, x))) dx d\tau \\
& + \int_{\sigma}^{\sigma+|h|} \int_{\{x: x \in Q, w_t(\tau, x) < 0\}} M_{\varphi} \left(- \left| \frac{w(\tau+h, x) - w(\tau, x)}{h} \right| \right) dx d\tau \\
& \leq \int_{\sigma}^{\sigma+|h|} \int_Q N_{\varphi}(\varphi(w_t(\tau, x))) dx d\tau + \int_{\sigma}^{\sigma+|h|} \int_Q \int_0^1 M_{\varphi}(|w_t(\tau+h\mu, x)|) d\mu dx d\tau \\
& + \int_{\sigma}^{\sigma+|h|} \int_Q \int_0^1 M_{\varphi}(-|w_t(\tau+h\mu, x)|) d\mu dx d\tau \leq \int_{\sigma}^{\sigma+|h|} \int_Q \varphi(w_t(\tau, x)) w_t(\tau, x) dx d\tau \\
& + \int_0^1 \int_{\sigma+h\mu}^{\sigma+h\mu+|h|} \int_Q (\varphi(|w_t(\tau, x)|) - \varphi(-|w_t(\tau, x)|)) |w_t(\tau, x)| dx d\tau d\mu.
\end{aligned}$$

Since, by the conditions on φ ,

$$\varphi(|w_t(\tau, x)|) - \varphi(-|w_t(\tau, x)|) = |\varphi(w_t(\tau, x))| + |\varphi(-w_t(\tau, x))|$$

from the above inequality, it follows that

$$\begin{aligned}
& \int_{\sigma}^{\sigma+|h|} \int_Q \left| \varphi(w_t(\tau, x)) \frac{w(\tau+h, x) - w(\tau, x)}{h} \right| dx d\tau \leq \int_{\sigma}^{\sigma+h} \int_Q \varphi(w_t(\tau, x)) w_t(\tau, x) dx d\tau \\
& + \int_0^1 \int_{\sigma+h\mu}^{\sigma+h\mu+|h|} \int_Q (|\varphi(w_t(\tau, x))| + |\varphi(-w_t(\tau, x))|) |w_t(\tau, x)| dx d\tau d\mu.
\end{aligned}$$

By the absolutely continuity property of the Lebesgue integral, we have

$$\int_{\sigma}^{\sigma+|h|} \int_Q \varphi(w_t(\tau, x)) w_t(\tau, x) dx d\tau \rightarrow 0 \text{ as } h \rightarrow 0$$

and

$$\int_{\sigma+h\mu}^{\sigma+h\mu+|h|} \int_Q (|\varphi(w_t(\tau, x))| + |\varphi(-w_t(\tau, x))|) |w_t(\tau, x)| dx d\tau \rightarrow 0 \text{ as } h \rightarrow 0,$$

uniformly with respect to $\sigma \in [s, t]$ and $\mu \in [0, 1]$. These approximations, together with the last inequality, complete the proof. \square

Lemma A.6. *Let $Q \subset \mathbb{R}^N$ be a measurable set of finite measure and φ an increasing continuous function such that $\varphi(0) = 0$. Also assume that $w \in C([0, 1]; L^{\infty}(Q))$ and $w_t \in L^1(0, 1; L^1(Q))$. Then*

$$\begin{aligned}
& \int_0^1 \int_{\Omega} \varphi(w_t(t, x)) w(t, x) dx dt \geq - \left(\frac{1}{M} \|w(0)\|_{L^{\infty}(\Omega)} + 1 \right) \int_0^1 \int_Q \varphi(w_t(t, x)) w_t(t, x) dx dt \\
& - \|\varphi\|_{C[-M, M]} \|w(0)\|_{L^1(\Omega)} - \int_0^1 \int_Q |\varphi(-w_t(s, x))| |w_t(s, x)| ds dx, \tag{A.25}
\end{aligned}$$

for every $M > 0$.

Proof. By (A.19), we find

$$\begin{aligned}
\int_0^1 \int_Q \varphi(w_t(t, x)) w(t, x) dx dt &= \int_0^1 \int_Q \varphi(w_t(t, x)) w(0, x) dx dt \\
&\quad + \int_0^1 \int_Q \varphi(w_t(t, x)) \int_0^t w_t(s, x) ds dx dt \\
&\geq - \int_0^1 \int_Q |\varphi(w_t(t, x)) w(0, x)| dx dt \\
&\quad - \int_0^1 \int_Q N_\varphi(\varphi(w_t(t, x))) dx dt - \int_0^1 \int_Q M_\varphi \left(- \int_0^t w_t(s, x) ds \right) dx dt.
\end{aligned} \tag{A.26}$$

From the conditions on φ , it follows that

$$\begin{aligned}
\int_0^1 \int_Q |\varphi(w_t(t, x)) w(0, x)| dx dt &= \int_0^1 \int_{\{x: x \in Q, |w_t(t, x)| \leq M\}} |\varphi(w_t(t, x)) w(0, x)| dx dt \\
&\quad + \int_0^1 \int_{\{x: x \in Q, |w_t(t, x)| > M\}} |\varphi(w_t(t, x)) w(0, x)| dx dt \leq \|\varphi\|_{C[-M, M]} \|w(0)\|_{L^1(\Omega)} \\
&\quad + \frac{1}{M} \|w(0)\|_{L^\infty(\Omega)} \int_0^1 \int_Q \varphi(w_t(t, x)) w_t(t, x) dx dt
\end{aligned} \tag{A.27}$$

and

$$\begin{aligned}
\int_0^1 \int_Q N_\varphi(\varphi(w_t(t, x))) dx dt &\leq \int_0^1 \int_Q \varphi^{-1}(\varphi(w_t(t, x))) \varphi(w_t(t, x)) dx dt \\
&= \int_0^1 \int_Q \varphi(w_t(t, x)) w_t(t, x) dx dt.
\end{aligned} \tag{A.28}$$

Also by Jensen's inequality and convexity of M_φ , we have

$$\begin{aligned}
\int_0^1 \int_Q M_\varphi \left(- \int_0^t w_t(s, x) ds \right) dx dt &= \int_0^1 \int_Q M_\varphi \left(t \frac{1}{t} \int_0^t -w_t(s, x) ds \right) dx dt \\
&\leq \int_0^1 \int_Q t M_\varphi \left(\frac{1}{t} \int_0^t -w_t(s, x) ds \right) dx dt \leq \int_0^1 \int_Q \int_0^t M_\varphi(-w_t(s, x)) ds dx dt \\
&\quad - \int_0^1 \int_Q \int_0^t \varphi(-w_t(s, x)) w_t(s, x) ds dx dt = \int_0^1 \int_Q \int_0^t |\varphi(-w_t(s, x))| |w_t(s, x)| ds dx dt \\
&\leq \int_0^1 \int_Q |\varphi(-w_t(s, x))| |w_t(s, x)| ds dx.
\end{aligned} \tag{A.29}$$

Thus, taking into account (A.27)-(A.29) in (A.26), we get (A.25). \square

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, HACETTEPE UNIVERSITY, BEYTEPE 06800, ANKARA, TURKEY

E-mail address: azer@hacettepe.edu.tr; azer_khan@yahoo.com